.:: شرح مادة ::.

(CALCULUS 2)

MATH-132

(CH 8.7 → CH 10.9)

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تلخيص المادة من محاضرات:

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* Ch(8.7):- Improper Integral:-

Improper Integral

Type I

$$\int_a^{\infty} f(x)\,dx,\ \int_{-\infty}^{b} f(x)\,dx,\ \int_{-\infty}^{\infty} f(x)\,dx$$

Type II

$$\int_{\boxed{a}}^{\boxed{b}} f(x)\,dx$$

such as $f(x)$ is discontinuous on $[a,b]$.

* (Type I, II) بعض الأمثلة للتمييز بين الأنواع *

* $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$, $\displaystyle\int_2^{\infty} \frac{dx}{x-1}$ (Type I)

* $\displaystyle\int_0^{2} \frac{dx}{(x-1)}$ (Type II)

  ↳ at $x=1$ is discontinuous
  غير متصل

* $\displaystyle\int_{-2}^{\infty} \frac{dx}{(x+2)}$ (Type I + Type II)

  $\infty \rightarrow$ (Type I)

  ↳ at $x=-2$ is discontious
  ↳ Type II

* $\displaystyle\int_{-\infty}^{4} \frac{dx}{(x^2)} \rightarrow$ at $x=0$ is discontious (Type I + Type II)

  ↳ Type II

  $-\infty$ ↳ Type I

* Type (I) :-

* f is cont. function on $[a, \infty)$

$$\int_{a}^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_{a}^{b} f(x)\,dx = \cdots$$

* f is cont. function on $(-\infty, a]$

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{b \to -\infty} \int_{b}^{a} f(x)\,dx = \cdots$$

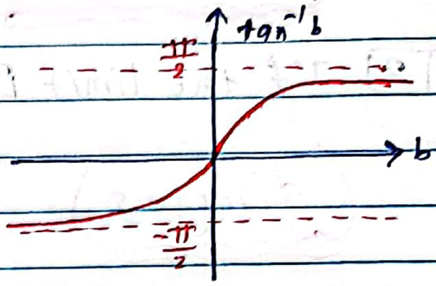
* f cont. function $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\boxed{c}} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

$$= \lim_{b \to -\infty} \int_{b}^{c} f(x)\,dx + \lim_{b \to \infty} \int_{c}^{b} f(x)\,dx$$

* Exp:- $\displaystyle\int_0^{\infty} \underbrace{\frac{dx}{x^2+1}}_{f(x)}$ (Type I)

$$= \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1}(x) \Big|_0^b$$



$$= \lim_{b\to\infty} \left[\tan^{-1}(b) - \underbrace{\cancel{\tan^{-1}(0)}}_{0} \right]$$

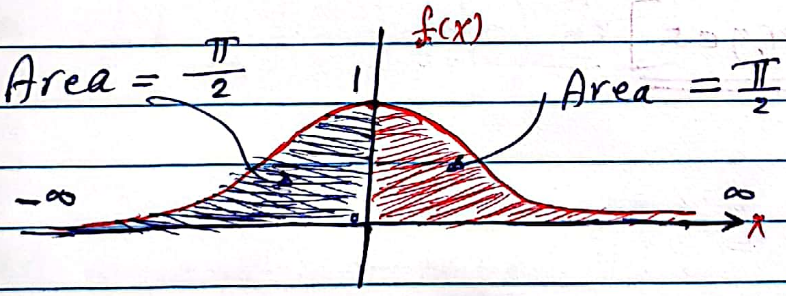
$$= \boxed{\frac{\pi}{2}} \ \#$$

* Exp:- $\displaystyle\int_{-\infty}^{0} \frac{dx}{x^2+1} = \boxed{\frac{\pi}{2}} \ \#$

* Exp:- $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_{0}^{\infty} \frac{dx}{x^2+1}$

من الرسمة تم إيجادهم

$$= \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$



[Converges]

* **Remark:-**

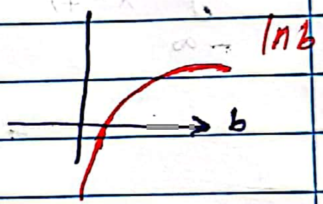
1. If the limit is finit (exists) then the Improper Integral (Converges)

2. If the limit DNE (or Infinit) then the Improper Integral (diverges)

* **Exp:-** $\displaystyle\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x}$

$$= \lim_{b\to\infty} \ln|x| \Big|_1^b = \lim \left[\ln|b| - \underbrace{\ln(1)}_{\boxed{0}}\right]$$

$$= \lim_{b\to\infty} \ln b = \boxed{\infty}$$



[diverges]

* Exp :- $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^2-1}\,dx$

$$= \lim_{b\to-\infty} \int_{b}^{-2} \frac{2}{x^2-1}\,dx$$

بطريقة الكسور الجزئية

$$\frac{2}{x^2-1} = \frac{2}{(x+1)(x-1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$A = \frac{2}{\boxed{1}+1} = \boxed{1} \checkmark$$

$$B = \frac{2}{\boxed{-1}-1} = \boxed{-1} \checkmark$$

$$= \lim_{b\to-\infty} \int_{b}^{-2} \left(\frac{A}{x-1} + \frac{B}{x+1}\right) dx$$

$$= \lim_{b\to-\infty} \int_{b}^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{b\to-\infty} \Big[\ln|x-1| - \ln|x+1|\Big]\Big|_{b}^{-2}$$

$$= \lim_{b\to-\infty} \ln\left|\frac{x-1}{x+1}\right|\Big|_{b}^{-2} = \lim_{b\to-\infty}\left[\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right]$$

$$= \ln 3 - \lim_{b \to -\infty} \ln \left| \frac{b-1}{b+1} \right|$$

$$= \ln 3 - \ln \left(\lim_{b \to -\infty} \frac{b-1}{b+1} \right) \to \frac{-\infty}{-\infty}, \frac{\infty}{\infty}$$

→ L'Hopital Rule

$$= \ln 3 - \cancel{\ln 1} \to \text{zero}$$

$$= \boxed{\ln 3} \Rightarrow \text{is finit}$$

↳ this is Converges to $\ln 3$

Exp:-

$$\int_{1}^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} \to \text{converges}, & \text{if } p > 1 \\ \infty \to \text{diverges} & \text{if } p \leq 1 \end{cases}$$

* this is only $1 \to \infty \Rightarrow$ (Type I)

*Exp**:-

$$\int_{0}^{1} \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} \to \text{converges}, & \text{if } p < 1 \\ \infty \to \text{diverges}, & \text{if } p \geq 1 \end{cases}$$

* this is only $0 \to 1 \Rightarrow$ (Type II)

* Exp:- $\boxed{1}$ $\int_1^\infty \frac{dx}{x^3} = \frac{1}{\boxed{3}-1} = \boxed{\frac{1}{2}}$ ~~this is finite~~ ~~since $p>1$ by Exp*~~

*this is (Type I) $\Rightarrow$ (Converges) #

$\boxed{2}$ $\int_1^\infty \frac{dx}{\sqrt{x}} = \boxed{\infty}$ since $p = \frac{2}{3} < 1$ by Exp*

*this is (Type I) $\Rightarrow$ (diverges) #

$\boxed{3}$ $\int_1^\infty \frac{dx}{x^{\frac{2}{3}}} = \boxed{\infty}$ since $p = \frac{1}{2} < 1$ by Exp*

*this is (Type I) $\Rightarrow$ (diverges) #

$\boxed{4}$ $\int_0^1 \frac{dx}{\sqrt{x}} = \frac{1}{1-\frac{1}{2}}$ $\left(= \frac{1}{\frac{1}{2}} = \right)$ $\boxed{2}$ this is finite since $p<1$, by Exp**

* this is (Type II) $\Rightarrow$ (Converges) #

* Type (II):-

* If f is discontinous at $\boxed{a}$, then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$

له الحد السفلي

* If f is discontinous at $\boxed{b}$, then $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$

له الحد العلوي

* If f is discontinous at $\boxed{c}$, where $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

نقطة داخلية يجب أن نفصل

* **Exp:-** $\int_0^4 \frac{dx}{\sqrt{4-x}}$ (Type II)

* at $\boxed{x=4}$ the function is discontinous. [المشكلة في الحد العلوي]

$$= \lim_{c \to 4^-} 2\int_0^c \frac{dx}{2\sqrt{4-x}} = \lim_{c \to 4^-} (-2)\sqrt{4-x}\,\Big|_0^c$$

$$= \lim_{c \to 4^-} -2\left[\sqrt{4-c} - \sqrt{4-0}\right]$$

$$= \lim_{c \to 4^-} -2\left[\sqrt{4-c} - 2\right] = \lim_{c \to 4^-} \left[-2\sqrt{4-c} + 4\right]$$

$$= \cancel{-2\sqrt{4-4}} + 4 = \boxed{4}$$

Converges #

*Exp: $\int_0^1 \frac{dx}{\sqrt{x}}$ (Type II)

* at $\boxed{x=0}$ the function is discontinous. [المشكلة في الحد السفلي]

$$= \lim_{c\to 0^+} 2\int_c^1 \frac{dx}{2\sqrt{x}} = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^1$$

$$= \lim_{c\to 0^+} \left(2\sqrt{1} - 2\sqrt{c}\right)$$

$$= \boxed{2} \checkmark \ \#$$

*. Exp: Q. 15

$$\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\, d\theta \quad (\text{Type II})$$

* at $\boxed{\theta=0}$ the function is discontinous. [المشكلة في الحد السفلي]

$\hookrightarrow \theta^2 + 2\theta = 0 \Rightarrow \theta(\theta+2) = 0$

$\boxed{\theta = 0} \checkmark$, $\boxed{\theta = -2}$ X ليست من ضمن الفترة

* $u = \theta^2 + 2\theta$

$du = (2\theta + 2)\, d\theta$

$\frac{du}{2} = (\theta+1)\, d\theta$

$$= \lim_{c\to 0^+} \int_c^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\, d\theta = \lim_{c\to 0^+} \int_c^1 \frac{\frac{du}{2}}{\sqrt{u}}$$

$$= \lim_{c\to 0^+} \int_c^1 \frac{du}{2\sqrt{u}} = \lim_{c\to 0^+} \sqrt{u}\Big|_c^1$$

$$= \lim_{c \to 0^+} \sqrt{\theta^2 + 2\theta} \Big|_c^1$$

$$= \lim_{c \to 0^+} \sqrt{1+2} - \sqrt{c^2 + 2c}$$

$$= \sqrt{3} - \sqrt{0} = \boxed{\sqrt{3}}$$ #✓

Converges #✓

* Exp:- $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$ (Q.18)

$\infty \to$ Type I

$1 \to$ Type II

* at $x=1$ the function is dis continous. هنا نفصل إلى تكاملين

$$= \underbrace{\int_1^2 \frac{dx}{x\sqrt{x^2-1}}}_{\text{Type II}} + \underbrace{\int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}}}_{\text{Type I}}$$

(هنا نختار أي رقم لكي نفصل لكن يجب أن يكون من ضمن الفترة ولا يصفر التكامل)

وهنا تم إختيار $\boxed{x=2}$ ✓

$$= \lim_{c \to 1^+} \int_c^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{b \to \infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}$$

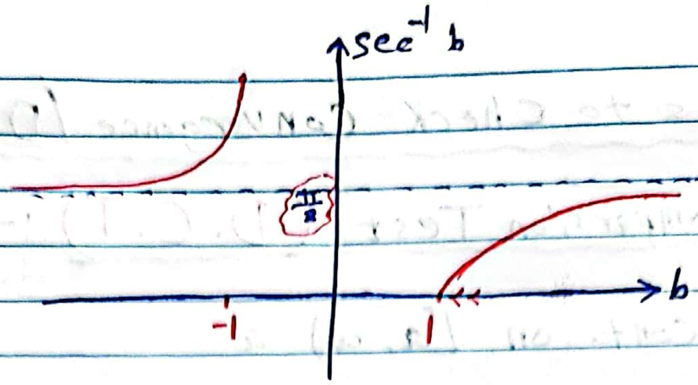
$$= \lim_{c \to 1^+} \sec^{-1}|x| \Big|_c^2 + \lim_{b \to \infty} \sec^{-1}|x| \Big|_2^b$$

$$= \lim_{c \to 1^+} \left[\cancel{\sec^{-1} 2} - \sec^{-1} c\right] + \lim_{b \to \infty} \left[\sec^{-1} b - \cancel{\sec^{-1}(2)}\right]$$

تم الغاءهم من البداية.

$$= \lim_{c \to 1^+} -\sec^{-1} c + \lim_{b \to \infty} \sec^{-1} b = 0 + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$ (Converges) #✓

CamScanner



* **Exp:-** $\displaystyle\int_0^\infty \frac{16\tan^{-1}(x)}{1+x^2}\,dx$ (Type I)

$$= \lim_{b\to\infty}\int_0^b \frac{16\overbrace{\tan^{-1}(x)}^{u}\,dx}{\underbrace{1+x^2}_{16du}}$$

$$\boxed{\begin{aligned} u &= \tan^{-1}(x) \\ 16du &= \frac{16\,dx}{1+x^2} \end{aligned}}$$

$$= \lim_{b\to\infty}\int_0^b 16u\,du = \lim_{b\to\infty} 8u^2\Big|_0^b$$

$$= \lim_{b\to\infty} 8\left(\tan^{-1}(x)\right)^2\Big|_0^b = 8\lim_{b\to\infty}\left(\left(\tan^{-1}(b)\right)^2 - \cancel{\left(\tan^{-1}(0)\right)^2}\right) \quad \boxed{0}$$

$$= 8\lim_{b\to\infty}\left(\tan^{-1}(b)\right)^2$$

$$= 8\left(\frac{\pi}{2}\right)^2 = 8\left(\frac{\pi^2}{4}\right)$$

$$= \boxed{2\pi^2}\ \#$$ Converges #

* Two Tests to check Convergence / Divergence :-

[1] Direct Comparsion Test (D.C.T) :-

f, g are cont. on $[a, \infty)$

$$0 \leq f(x) \leq g(x) \quad \forall x \in [a, \infty) \text{ then}$$

• If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ converge
الاقتران الكبير

• If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges
الاقتران الصغير

*** Exp:- Check the Convergence / Divergence :-**

[1] $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

Converges

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$$

$$* \int_1^\infty \frac{dx}{x^2} = \frac{1}{\boxed{2}-1} = \boxed{1}$$

$\hookrightarrow \int_1^\infty \frac{\sin^2 x}{x^2}$ Converges by D.C.T

* EXP:- $\int_0^{\pi} \frac{dx}{\sqrt{t} + \sin t}$ (Type II)

diverges $\qquad\qquad$ Converges

$$\bigcirc < \frac{1}{\sqrt{t} + \sin t} \le \frac{1}{\sqrt{t}}$$

* You need to Check $\frac{1}{\sqrt{t}}$

$$\int_0^{\pi} \frac{dt}{\sqrt{t}} = \lim_{b \to 0^+} \int_b^{\pi} \frac{2\,dt}{2\sqrt{t}} = \lim_{b \to 0^+} 2\sqrt{t} \Big|_b^{\pi}$$

$$= \lim_{b \to 0^+} \left(2\sqrt{\pi} - \cancel{2\sqrt{b}}\right) = 2\sqrt{\pi}$$

* $\int_0^{\pi} \frac{dx}{\sqrt{x} + \sin x}$ converges # by D.C.T

* Exp:- $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^2-0.1}}$

$x^2-0.1 \Rightarrow x^2 = 0.1$

↳ $x = \mp\sqrt{0.1}$

$\notin [1,\infty)$, so this not type II.

div. Conv.

$$\frac{1}{\sqrt{x^2}} \leq \frac{1}{\sqrt{x^2-0.1}} \leq \quad$$

↳ You need to check $\dfrac{1}{\sqrt{x^2}} = \boxed{\dfrac{1}{x}}$

$\displaystyle\int_1^\infty \frac{1}{x}\,dx \Rightarrow$ Div by Exp*

↳ $\displaystyle\int \frac{dx}{\sqrt{x^2-0.1}} \Rightarrow$ div by D.C.T

* Exp:- Q.(7)

$$\int_1^{\infty} \frac{dx}{x^3+1}$$

Conv. by Exp*

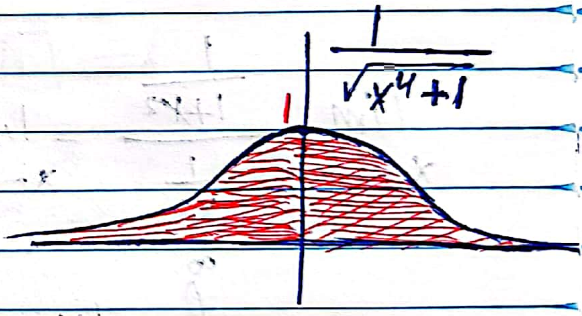
$$\ll \frac{1}{x^3+1} \ll \frac{1}{x^3}$$

$\hookrightarrow \int_1^{\infty} \frac{1}{x^3+1}\,dx$ conv. by $\underline{\underline{D.C.T}}$

* Exp: $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ (Type I)

الاقتران زوجي (even)

$$= \int_{-\infty}^{\boxed{0}} \frac{dx}{\sqrt{x^4+1}} + \int_{\boxed{0}}^{\infty} \frac{dx}{\sqrt{x^4+1}}$$



$$= 2\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}}$$

Conv. by Exp*

$$\frac{1}{\sqrt{x^4+1}} \ll \frac{1}{\sqrt{x^4}}$$

$\hookrightarrow$ Hence, $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$ Conv. by DCT $\frac{1}{x^2}$

2 Limit Comparsion Test (L.C.T) :-

f, g are positive, Cont. on $[a, \infty)$ and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \quad \text{where } 0 < L < \infty \text{ then}$$

$\int_a^\infty f(x)\,dx$, and $\int_a^\infty g(x)\,dx$, both are diverging

or both are Converging

* Exp:- Check Conv. / Div :-

Conv. by Exp ??

1. $\int_1^\infty \frac{dx}{1+x^2}$ * $f = \frac{1}{1+x^2}$ معطى بالسؤال $\longrightarrow g = \frac{1}{x^2}$

$$\lim_{x \to \infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{x^2}{1+x^2} = \boxed{1} \rightarrow L .$$

this is finite

So, $\int_1^\infty \frac{dx}{1+x^2}$ Conv by L.C.T

$\boxed{2}$ $\int_2^\infty \frac{dx}{\sqrt{x-1}}$

$* f = \frac{1}{\sqrt{x-1}}$

div ✓ ??

$* g = \frac{1}{\sqrt{x}}$

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_2^b \frac{2\,dx}{2\sqrt{x}} = \lim_{b\to\infty} 2\sqrt{x}\,\Big|_2^b$$

$$= \lim_{b\to\infty} 2\sqrt{b} - 2\sqrt{2} = \boxed{\infty} \to \text{div.}$$

$$\lim_{x\to\infty} \frac{f}{g} = \lim \frac{\frac{1}{\sqrt{x-1}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x-1}}$$

$$= \boxed{1} = L$$

$\int_2^\infty \frac{dx}{\sqrt{x-1}}$ div. by L.C.T #✓

*Ch (10.1) :- Sequences:-

* A sequences is a list of numbers "$a_1, a_2, a_3 ---- a_n$"

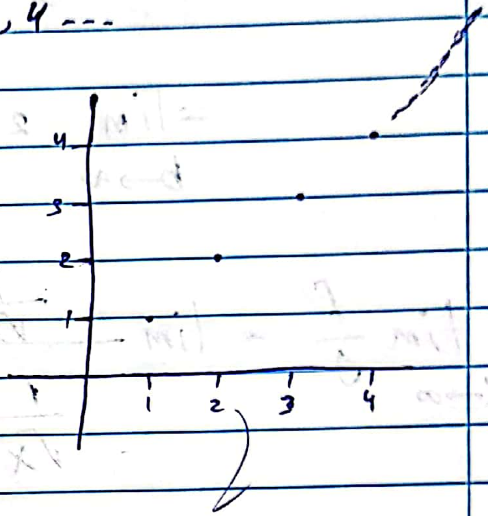
* Sequence → finite / → infinite

Converge   diverge

*Exp:- $\boxed{1}$ $a_n = \sqrt{n}$, $n = 1, 2, 3, 4 ---$

$n=1 \Rightarrow a_1 = \sqrt{1} = \boxed{1}$

$n=2 \Rightarrow a_2 = \sqrt{2} = \boxed{2}$



$a_n = \sqrt{n}$

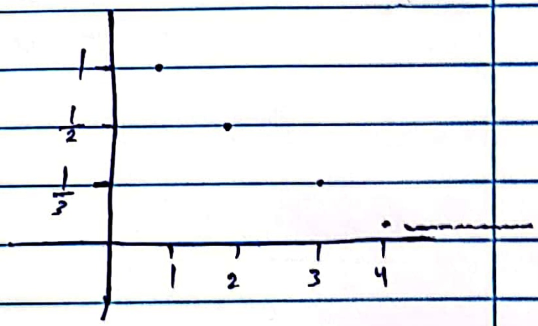
$\lim\limits_{n \to \infty} \sqrt{n} = \infty \Rightarrow a_n = \sqrt{n}$ div. # ($a_n \to \infty$ as $n \to \infty$)

$\boxed{2}$ $b_n = \frac{1}{n}$, $n = 1, 2, 3, -----$

$n=1 \Rightarrow b_1 = 1$

$n=2 \Rightarrow b = \frac{1}{2}$

$n=3 \Rightarrow b = \frac{1}{3}$



$\lim\limits_{n \to \infty} \frac{1}{n} = 0 \Rightarrow b_n = \frac{1}{n}$ Converg. # ($b_n \to 0$ as $n \to \infty$

$\boxed{3}$ $C_n = (-1)^n \frac{1}{n}$ [Alternating sequence]

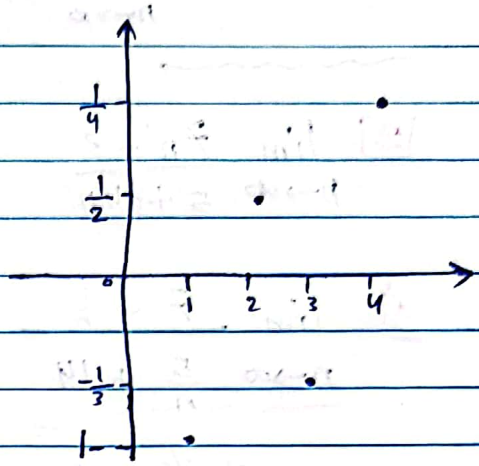
$n = 1, 2, 3, 4, \dots$

$n=1 \Rightarrow C_1 = (-1)^1 \frac{1}{1} = \boxed{-1}$

$n=2 \Rightarrow C_2 = (-1)^2 \frac{1}{2} = \boxed{\frac{1}{2}}$

$n=3 \Rightarrow C_3 = (-1)^3 \frac{1}{3} = \boxed{\frac{-1}{3}}$

$n=4 \Rightarrow C_4 = (-1)^4 \frac{1}{4} = \boxed{\frac{1}{4}}$



Th:- $\lim\limits_{n\to\infty} a_n = A$, $\lim\limits_{n\to\infty} b_n = B$ Then

A: number
$a_n \to A$ as $n \to \infty$

B: number
$b_n \to B$ as $n \to \infty$

a_n, b_n } conv.

$\boxed{1}$ $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$

$\boxed{2}$ $\lim\limits_{n\to\infty} (a_n - b_n) = A - B$

$\boxed{3}$ $\lim\limits_{n\to\infty} K a_n = K \cdot A$

K: Constant

$\boxed{4}$ $\lim\limits_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

$\boxed{5}$ $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, $B \neq 0$.

*Exp: [1] $\lim_{n\to\infty} \frac{-\sqrt{5}}{n}$

$= -\sqrt{5} \lim_{n\to\infty} \frac{1}{n} = -\sqrt{5}\,(0) = \boxed{0}$ # ✓

[2] $\lim_{n\to\infty} \frac{7n^1-3}{5+14n^1} = \boxed{\frac{7}{14}}$

أو $\lim_{n\to\infty} \frac{7-\frac{3}{n}}{\frac{5}{n}+14} = \frac{7-0}{0+14} = \boxed{\frac{7}{14}}$

قسمنا على n

[3] $\lim_{n\to\infty} \frac{7n-3}{5+14n^2} = \boxed{0}$ ✓

أو $\lim_{n\to\infty} \frac{\frac{7}{n}-\frac{3}{n^2}}{\frac{5}{n^2}+14} = \frac{0-0}{0+14} = \boxed{0}$ ✓

[4] $\lim_{n\to\infty} \frac{7n^3-3}{5+14n} = \boxed{\infty}$ # ✓

أو

$\lim \frac{7n^2-\frac{3}{n}}{\frac{5}{n}+14} = \frac{\infty-0}{0+14} = \boxed{\infty}$ # ✓

* ملاحظات:- [1] درجة البسط = درجة المقام $\xLeftarrow{\text{الجواب}}$ معامل البسط / معامل المقام.

[2] درجة البسط < درجة المقام $\xLeftarrow{\text{الجواب}}$ $\boxed{\text{zero}}$.

[3] درجة البسط > درجة المقام $\xLeftarrow{\text{الجواب}}$ $\boxed{\infty}$.

* Sandwich Th:-

If $a_n \leq b_n \leq C_n$ for all (n)

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} C_n = \boxed{L}$, L finite

Then $\lim_{n\to\infty} b_n = \boxed{L}$

* Exp:- Check Conv. / Div. for

[1] $\lim_{n\to\infty} \frac{\sin n}{n}$

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

$$\underbrace{\lim_{n\to\infty} -\frac{1}{n}}_{0} \leq \lim_{n\to\infty} \frac{\sin n}{n} \leq \lim_{n\to\infty} \frac{1}{n} \to 0$$

zero by Sandwich Th

[2] Q. 46) $\lim_{n\to\infty} \frac{\sin^2 n}{2^n}$

$$\frac{0}{2^n} \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$$

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{\sin^2 n}{2^n} \leq \lim_{n\to\infty} \frac{1}{2^n} \to \text{zero}$$

↳ zero   ↳ zero by sandwich Th

* **Note:-**

$$a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \Rightarrow \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$$

* Such that :- $x \in (-1,1) \Rightarrow \lim_{n\to\infty} x^n = 0$, $|x|<1$

$\boxed{3}$ $\lim_{n\to\infty} (-1)^n \frac{1}{n}$

$$\frac{-1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$$

$$\underbrace{\lim_{n\to\infty} \frac{-1}{n}}_{\text{zero}} \leq \underbrace{\lim_{n\to\infty} (-1)^n \frac{1}{n}}_{\text{zero}} \leq \underbrace{\lim_{n\to\infty} \frac{1}{n}}_{\text{zero}}$$

by sandwich Th.

---

* **Th:-**

① $\lim_{n\to\infty} \frac{\ln n}{n} = \boxed{0}$

لوبيتال $\frac{\infty}{\infty}$ $\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \boxed{0}$ ✓

* **Exp:-** $\lim_{n\to\infty} \frac{\ln n^3}{3n} = \lim_{n\to\infty} \frac{\cancel{3}\ln n}{\cancel{3}n} = \lim_{n\to\infty} \frac{\ln n}{n} = \boxed{0}$ ✓

② $\lim_{n\to\infty} \sqrt[n]{n} = \boxed{1}$

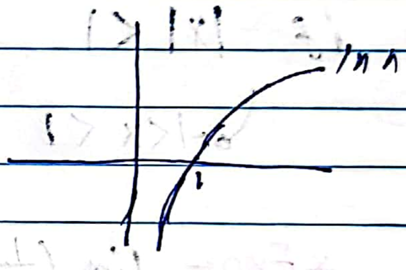
الحل

$$\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \infty^0 \quad \text{(indetermined Power)}$$

$$= \lim_{n\to\infty} e^{\ln\sqrt[n]{n}} = \lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\ln n}{n}}$$

$$= e^{\lim_{n\to\infty}\frac{\ln n}{n}} = e^0 = \boxed{1}$$

* Expr (Q. 59) $\lim_{n\to\infty} \frac{\ln n}{\sqrt[n]{n}}$



$$= \frac{\lim_{n\to\infty} \ln(n)}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{\infty}{1} = \boxed{\infty} \checkmark$$

* Expr $\lim_{n\to\infty} \sqrt[n]{n^3} = \lim_{n\to\infty} (n^3)^{\frac{1}{n}} = \lim_{n\to\infty} \left(n^{\frac{1}{n}}\right)^3$

$$= (1)^3 = \boxed{1} \checkmark$$ #

* Expr $\lim_{n\to\infty} \sqrt[n]{\pi n}$

$$= \lim_{n\to\infty} (\pi)^{\frac{1}{n}} \cdot (n)^{\frac{1}{n}} = \underbrace{\lim_{n\to\infty} \pi^{\frac{1}{n}}}_{\boxed{1}} \cdot \underbrace{\lim_{n\to\infty} \sqrt[n]{n}}_{\boxed{1}} = \boxed{1} \checkmark$$ #

③ $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = \boxed{1}$, $x > 0$

الحل: $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = \lim\limits_{n\to\infty} e^{\ln x^{\frac{1}{n}}} = \lim\limits_{n\to\infty} e^{\frac{\ln x}{n}}$

تم إخراجها لأنها ليست تعتمد على (n). ← $= e^{(\ln x) \cdot \lim\limits_{n\to\infty} \frac{1}{n}}$

$= e^{(\ln x)\cdot(0)} = e^0 = \boxed{1}$ ✓

④ if $|x| < 1 \Longrightarrow \lim\limits_{n\to\infty} x^n = \boxed{0}$

$\hookrightarrow -1 < x < 1$

*Exp:- $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = \boxed{0}$, $\lim\limits_{n\to\infty} \left(\frac{-2}{3}\right)^n = \boxed{0}$

$\lim\limits_{n\to\infty} 2^n = \boxed{\infty}$, $\lim\limits_{n\to\infty} \left(\frac{3}{2}\right)^n = \boxed{\infty}$

*Exp:- $\lim\limits_{n\to\infty} \frac{\pi^{-n}}{e^{-n}} = \lim\limits_{n\to\infty} \left(\frac{\pi}{e}\right)^{-n} = \lim\limits_{n\to\infty} \left(\frac{e}{\pi}\right)^n$ <1

$= \boxed{0}$

*Exp:- $\lim\limits_{n\to\infty} \left(\frac{\pi}{e}\right)^n = \boxed{\infty}$

[illegible]

(5) $\lim\limits_{n\to\infty}\left(1 + \frac{x}{n}\right)^{n} = e^{x}$

أصلها $\lim\limits_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^{n}} = \lim\limits_{n\to\infty} e^{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}}$

$= e^{\lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}}$ $\quad \frac{0}{0}$ L'Hopital Rule

$= e^{\lim\limits_{n\to\infty}\frac{\frac{\frac{-x}{n^2}}{1+\frac{x}{n}}}{\frac{-1}{n^2}}}$

$= e^{\lim\limits_{n\to\infty}\frac{x}{1+\frac{x}{n}}} = e^{\lim\limits_{n\to\infty}\frac{x}{1+0}} = \boxed{e^{x}}$

* Exp: [1] $\lim\limits_{n\to\infty}\left(1+\frac{3}{n}\right)^{n} = \boxed{e^{3}}$

[2] $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} = e^{-1} = \boxed{\frac{1}{e}}$

ندخلها

* Q. 68) $\lim\limits_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^{n} = \ln\left(\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right) \to e^{1} = e$

$= \ln e = \boxed{1}$ # ✓

* Q.84) $\lim_{n\to\infty} \sqrt[n]{n^2+n}$

$$= \lim_{n\to\infty} e^{\ln (n^2+n)^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} e^{\frac{\ln(n^2+n)}{n}} = e^{\lim_{n\to\infty} \frac{\ln(n^2+n)}{n}} = \frac{\infty}{\infty}$$ L'Hôpital Rule

$$= e^{\lim_{n\to\infty} \frac{\frac{2n+1}{n^2+n}}{1}} \to 0$$

$$= e^0 = \boxed{1}$$

$\boxed{6}$ $\lim_{n\to\infty} \frac{x^n}{n!} = \boxed{0}$ for any x "Taylor series"

* Exp: $\lim_{n\to\infty} \frac{2^n}{n!} = 0$, $\lim_{n\to\infty} \frac{(\frac{1}{2})^n}{n!} = 0$

* Exp: $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty} \left(\frac{n-1+2}{n-1}\right)^n$

$$= \lim_{n\to\infty} \left(1 + \frac{2}{n-1}\right)^n \qquad \left\{ \begin{array}{l} u = n-1 \\ \boxed{u+1 = n} \end{array} \right.$$

$$= \lim_{u\to\infty} \left(1 + \frac{2}{u}\right)^{u+1} = \lim_{u\to\infty} \left(1+\frac{2}{u}\right)\left(1+\frac{2}{u}\right)^u$$

$$= (1+0)(e^2) = \boxed{e^2} \#$$

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*Exp:- Find the n^{th} term of the following sequences:-

$\boxed{1}$ $1, -4, 9, -16, 25, \ldots\ldots ??$

1^{st} term, 2^{nd} term, 3^{rd} term, $n=4$, $n=5 \ldots$, n^{th} term

$\hookrightarrow a_n = (-1)^{n+1} \, n^2$

لإشارة السالب علشان مرة القوة تكون فردي ومرة تكون زوجي لذلك إذا كانت القوة زوجية يكون موجب وإذا كانت فردية يكون سالب

علشان الأرقام مربعة يعني

$(1)^2 = 1$

$(2)^2 = 4$

$(3)^2 = 9$

$\vdots$

** هذا الحل اذا كانت (n) تبدأ من $\boxed{1}$

← لكن اذا بدأت (n) من الـ $\boxed{\text{Zero}}$ ⇐ $\hookrightarrow a_n = (-1)^n (n+1)^2$

$\boxed{2}$ $0, 3, 8, 15, 24, \ldots\ldots ??$ (a_n)

$\hookrightarrow a_n = (n^2 - 1)$, *such that : $n = 1, 2, 3, 4 \ldots$

$\boxed{3}$ $-3, -2, -1, 0, 1, 2, 3, \ldots$

$\hookrightarrow a_n = n - 4$, *such that : $n = 1, 2, 3, \ldots$

OR $\hookrightarrow a_n = n - 3$, *such that: $n = 0, 1, 2, 3 \ldots$

* Recursive Sequence :-

وفكرتها إنه اذا بدي الحد a_{100}
لازم أكون عارف قيمة الحد a_{99}.

* Exp :- $a_1 = 1 \quad , \quad a_{n+1} = \frac{1}{2} a_n$

Assume this sequence Converges, find it's limit, Find $\lim_{n\to\infty} a_n$

$$a_1 = 1$$

$$a_2 = a_{\boxed{1}+1} = \frac{1}{2} a_1 = \frac{1}{2}(1) = \boxed{\frac{1}{2}}$$

$$a_3 = a_{\boxed{2}+1} = \frac{1}{2} a_2 = \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$$

$$a_4 = a_{\boxed{3}+1} = \frac{1}{2} a_3 = \frac{1}{2} \cdot \frac{1}{4} = \boxed{\frac{1}{8}}$$

$\to a_n = \left(\frac{1}{2}\right)^{n-1}$, *such that $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n} \cdot \left(\frac{1}{2}\right)^{-1}$$

$-1 < \frac{1}{2} < 1$

$$= \left(\frac{1}{2}\right)^{-1} \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n}$$

$$= (2)(0) = \boxed{0} \; \#$$

$\to$ Converges

* Exp: 3, 3, 3, 3, 3, ... Find C_n

$C_n = 3$, $n = 1, 2, \dots$

$\lim_{n\to\infty} C_n = \lim_{n\to\infty} 3 = \boxed{3}$

→ * Converges

* Def:

• A sequence $\{a_n\}$ is bounded from above if $\exists$ a number M, such that $a_n \leq M$ for all n.

→ Upper bound

يوجد

• A sequence $\{a_n\}$ is bounded from below if $\exists$ a number m, such that $a_n \geq m$ for all n.

→ lower bound

* Exp:

$\boxed{1}$ $a_n = 1, 2, 3, 4, 5, \dots$ non decreasing / monotonic

* M ?? a_n is not bounded from above.

greatest lower bound * $m = 1, 0, -1, -2, -25, \dots$ lower bounds

a_n is only bounded from below

→ a_n is not bounded

$\boxed{2}$ $b_n = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ non increasing / monotonic

* $M = 1, 2, 3, 4.5, e, 100, \dots$ Upper bound

→ least upper bound

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$*m = 0, -1, -2, -3 \ldots$ lower bound
↳ greatest lower bound

* b_n is bounded from below and bounded from above $\Rightarrow$ b_n is bounded

$\boxed{3}$ $C_n = 3, 3, 3, 3, \ldots$ / non increasing and non decreasing and monotonic

$*M = 3, 4.5, 8.1, 9, 17, \ldots$ Upper bound
↳ least upper bound

$*m = 3, 2, 1, 0, -1, \ldots$
↳ greatest lower bound

C_n is bounded from below and bounded from above $\Rightarrow C_n$ is bounded

- A sequence $\{a_n\}$ is bounded if it is bounded from above and is bounded from below.

- A sequence $\{a_n\}$ is not bounded if it is not bounded from above and is not bounded from below

*Def:

- A sequence $\{a_n\}$ is nondecreasing if $a_n \leq a_{n+1} \ \forall n$

  ↳ $a_1 \leq a_2 \leq a_3 \ldots$

- A sequence $\{a_n\}$ is non increasing if $a_n \geq a_{n+1} \ \forall n$

  $a_1 \geq a_2 \geq a_3 \ldots$

- A sequence $\{a_n\}$ is monotonic if it is either non decreasing or non increasing.

*Exp: $1, -1, 1, -1, 1, -1$

is not monotonic

*Th: If a sequence $\{a_n\}$ is both ① bounded ② monotonic

Then $\{a_n\}$ Converges

*Exp: $a_n = \frac{1}{n}$, $n = 1, 2, 3, ...$

$\boxed{1}$ Find m, M.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$

*$m = 0, -1, -2, -3, ...$ lower bounds

greatest lower bound

*$M = 1, 2, 3, ...$ upper bounds

lowest upper bound

$\boxed{2}$ Is a_n Monotonic?

* a_n is nonincreasing $\Rightarrow$ monotonic

$\boxed{3}$ Is a_n bounded?

Yes, since we found m, M.

$\boxed{4}$ Does a_n Converges? Yes, since $\{a_n\}$ monotonic and bounded

*Ch (10.2): Infinite Series:-

* An Infinite series is the sum of an infinite sequence of numbers $a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$

↳ * such that:-
- a_n : is the n^{th} term of the series.
- $S_1 = a_1$ is the 1^{st} partial sum of the series
- $S_2 = a_1 + a_2$ is the 2^{nd} partial sum of the series

$\vdots$

- $S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{K=1}^{n} a_K$ is the n^{th} partial sum of series.

* Test (1): " n^{th} partial sum Test "

Assume:- $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

↳ Find $S_n = \sum_{K=1}^{n} a_K = a_1 + a_2 + a_3 + \cdots a_n$.

- If $\lim_{n\to\infty} S_n = L$, then $\sum_{n=1}^{\infty} a_n$ Converges to L.
- If $\lim_{n\to\infty} S_n$ div, then $\sum_{n=1}^{\infty} a_n$ div.

* This Test Check Conv. and div

* **Exp:-** (Telescroping Series), Check for div. / conv.?

[1] $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

* $S_n = \left(\frac{1}{\sqrt{1}} - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right) + \dots + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right)$

$= 1 - \frac{1}{\sqrt{n+1}}$

* $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1 - \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 1 - 0$

$= \boxed{1} = L$

↳ Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$ Converges to 1 by n^{th} partial sum Test.

[2] Use n^{th} partial sum Test to check conv. / div.

Q.37) $\sum_{n=1}^{\infty}\left(\ln\sqrt{n+1} - \ln\sqrt{n}\right)$

* $S_n = \left(\cancel{\ln\sqrt{2}} - \ln\sqrt{1}\right) + \left(\cancel{\ln\sqrt{3}} - \cancel{\ln\sqrt{2}}\right) + \left(\ln\sqrt{n+1} - \cancel{\ln\sqrt{n}}\right)$

$= -\ln\sqrt{1} + \ln\sqrt{n+1} = -\underbrace{\cancel{\ln 1}}_{\text{zero}} + \ln\sqrt{n+1} = \boxed{\ln\sqrt{n+1}}$

* $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\ln\sqrt{n+1} = \ln\underbrace{\left(\lim_{n\to\infty}\sqrt{n+1}\right)}_{\infty} = \infty$

↳ Hence, $\sum_{n=1}^{\infty}\left(\ln\sqrt{n+1} - \ln\sqrt{n}\right)$ diverges by n^{th} partial sum Test.

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3 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

* Using Partial fraction in Math 1411

$$\frac{1}{n(n+1)} = \frac{A}{\underset{n=0}{n}} + \frac{B}{\underset{n=-1}{n+1}}$$

* $A = \frac{1}{0+1} = \boxed{1}$

* $B = \frac{1}{-1} = \boxed{-1}$

$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$$

$$= \boxed{\frac{1}{n} - \frac{1}{n+1}} \#$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

* $S_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$$= 1 - \frac{1}{n+1}$$

* $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - \lim \frac{1}{n+1} = 1 - 0$

$$= \boxed{1}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ Converges to 1 by n^{th} partial sum test

* Test (2):- n^{th} term test for div.

- If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ div.

- If $\lim_{n\to\infty} a_n$ fails to exist, then $\sum_{n=1}^{\infty} a_n$ div.
  (∞, $-\infty$, DNE).

* Exp:- Check for Conv. / Div

1) $\sum_{n=1}^{\infty} \frac{n+1}{n} \rightarrow a_n$

$$* \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{1} = \boxed{1} \neq 0$$

↳ Hence, $\sum_{n=1}^{\infty} \frac{n+1}{n}$ div. by n^{th} term test.

2) $\sum_{n=1}^{\infty} \sqrt{n}$

$$* \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n} = \boxed{\infty} \neq 0$$

↳ Hence, $\sum_{n=1}^{\infty} \sqrt{n}$ div by n^{th} term test.

$\boxed{3}$ Q.64) $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$

* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n + 4^n}{3^n + 4^n} = \lim_{n\to\infty} \frac{\frac{2^n}{4^n} + 1}{\frac{3^n}{4^n} + 1} = \lim \frac{(\frac{2}{4})^n + 1}{(\frac{3}{4})^n + 1}$

$= \frac{1 + \underbrace{\lim_{n\to\infty} (\frac{1}{2})^n}_{\text{zero}}}{1 + \underbrace{\lim_{n\to\infty} (\frac{3}{4})^n}_{\text{zero}}} = \boxed{1} \neq 0$

↳ Hence, $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$ div by n^{th} term test.

$\boxed{4}$ $\sum_{n=1}^{\infty} (-1)^{n+1}$

* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1} = \text{D.N.E}$

↳ Hence, $\sum_{n=1}^{\infty} (-1)^{n+1}$ div. by n^{th} term test.

$\sum_{n=1}^{\infty} (-1)^{n+1} = \cancel{1-1} + \cancel{1-1} + \cancel{1-1} \cdots = \boxed{0}$

OR

$\sum_{n=1}^{\infty} (-1)^{n+1} = 1 + (\cancel{-1+1}) + (\cancel{-1}+1) + \cdots = \boxed{1}$

$\Rightarrow$ D.N.E

* Harmonic Saries div $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$

$\hookrightarrow \sum_{n=1}^{\infty} \frac{1}{n} \geq \infty \Rightarrow$ div.

* Th:- • If $\lim_{n\to\infty} a_n = \boxed{0}$, then $\sum_{n=1}^{\infty}$ may div.
may conv.

• If $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{Conv.}}$, then $\lim_{n\to\infty} a_n =$ zero

• If $\lim_{n\to\infty} a_n = 0$, this does not mean $\sum_{n=1}^{\infty} a_n$ Conv.

$\hookrightarrow$ *Exp:- Harmonic Saries $\Rightarrow$ div

$\hookrightarrow$ but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = \boxed{0}$

* Th:- Assume $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, Then:-

$\boxed{1}$ $\sum_{n=1}^{\infty} (a_n \mp b_n) = A \mp B$

$\hookrightarrow$ The sum of two converged infinite series is Convergent
subtraction

$\boxed{2}$ $\sum_{n=1}^{\infty} K a_n = K \cdot A$

$\hookrightarrow$ Any constant multiple of convergent selres is convergent.

* Remarks:-

1. If $\sum a_n$ div. then $\sum K\, a_n$ div (constant $\neq 0$)

2. If $\sum a_n$ conv. and $\sum b_n$ div then

$$\sum (a_n + b_n) \text{ div} \quad \text{and} \quad \sum (a_n - b_n) \text{ div}$$

---

* Test(3):- Geometric Series Test:-

ممكن تكون ~

Conv. | div.

* Geometric Series has the form :-

$$\sum_{n=1}^{\infty} a r^{n-1} = \underbrace{a}_{1^{st}\text{ term}} + \underbrace{ar}_{2^{nd}\text{ term}} + ar^2 + ar^3 + ar^4 + \cdots$$

* such that:- r : ration (الثابت)

* $\dfrac{\text{الحد الجديد}}{\text{الحد اللي قبل}} = r \Rightarrow \dfrac{\cancel{a}r}{\cancel{a}} = r, \quad \dfrac{\cancel{ar}^{2}}{\cancel{ar}} = r, \quad \dfrac{ar^3}{ar^2} = r \cdots$

- $\sum_{n=1}^{\infty} a r^{n-1}$ conv. If $|r| < 1 \Rightarrow -1 < r < 1$

div ← [conv.] → div, r ; −1 , 1

↳ * sum $= \sum_{n=1}^{\infty} a r^{n-1} = \dfrac{a}{1-r}$

- $\sum_{n=1}^{\infty} a r^{n-1}$ div If $|r| \geq 1 \Rightarrow r \geq 1$ or $r \leq -1$

* Exp: Find sum of

$\boxed{1}$ $\underset{a}{1} + \frac{1}{2} + \frac{1}{4} + \dots + \left(\frac{1}{2}\right)^{n-1}$

* $\frac{\frac{1}{2}}{1} = \underset{r}{\boxed{\frac{1}{2}}}$, $\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \cdot 2 = \underset{r}{\boxed{\frac{1}{2}}}$

$\hookrightarrow r = \frac{1}{2} \in (-1,1)$

Geometric series $\Rightarrow$ Conv. since $r \in (-1,1)$.

* $\text{sum} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$ #.

$\hookrightarrow$ Hence, $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2 \Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ Converges to 2

$\boxed{2}$ $\underset{a}{1} - \frac{1}{3} + \frac{1}{9} + \dots + \left(-\frac{1}{3}\right)^{n-1} + \dots$

* $\frac{-\frac{1}{3}}{1} = \underset{r}{\boxed{-\frac{1}{3}}}$, $\frac{\frac{1}{9}}{-\frac{1}{3}} = \underset{r}{\boxed{-\frac{1}{3}}}$

$\hookrightarrow r = -\frac{1}{3} \in (-1,1)$

Geometric series $\Rightarrow$ Conv. since $r \in (-1,1)$

* $\text{sum} = \frac{a}{1-r} = \frac{1}{1+\frac{1}{3}} = \boxed{\frac{3}{4}}$ #

$\hookrightarrow$ Hence, $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = \boxed{\frac{3}{4}} \Rightarrow \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} \underset{\text{Conv.}}{\underset{\downarrow}{\text{to}}} \frac{3}{4}$

3 Check Conv. / div

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$$

$$= \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^3 + \ldots$$

$$* \, r = \frac{\left(\frac{4}{3}\right)^2}{\frac{4}{3}} = \boxed{\frac{4}{3}} > 1 \Rightarrow$$ This series is Geometric

↳ div

* Exp: (Q. 51)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$$, check Conv. / div.

$$↳ \; 3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n} = 3\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots\right]$$

↳ a

$$* \, r = \frac{-\frac{1}{4}}{\frac{1}{2}} = -\frac{1}{2}$$

↳ Geometric Series $\Rightarrow$ Conv. since $r = -\frac{1}{2} \in (-1, 1)$.

$$* \, \text{Sum} = 3\left[\frac{a}{1-r}\right] = 3\left[\frac{\frac{1}{2}}{1-\frac{-1}{2}}\right] = 3\left[\frac{\frac{1}{2}}{\frac{3}{2}}\right] = \frac{3}{3} = \boxed{1}$$

Hence, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n} = 1 \Rightarrow \sum (-1)^{n+1} \frac{3}{2^n}$ conv. to 1

*Exp.- Q. 75)

Consider $\sum_{n=0}^{\infty}(-1)^n (x+1)^n$, [1] Find x such that this series conv.
[2] Find it's sum.

$= 1 - (x+1) + (x+1)^2 - (x+1)^3 + (x+1)^4 + \dots$

1 ↳ a

$*r = \frac{-(x+1)}{1} \overset{??}{=} \frac{(x+1)^2}{-(x+1)} \overset{??}{=} \frac{-(x+1)^3}{(x+1)^2} = -(x+1)$

↳ This series is Geometric since r is constant.

[1] $|r| < 1 \Leftarrow$ Conv. بما إنه التسلسلية

↳ $|-(x+1)| < 1 \Rightarrow |x+1| < 1$

↳ $-1 < x+1 < 1$

$\boxed{-2 < x < 0}$ ✓

[2] $\text{sum} = \frac{a}{1-r} = \frac{1}{1-(-x+1)} = \frac{1}{1+x+1} = \boxed{\frac{1}{2+x}}$

↳ $\sum_{n=0}^{\infty}(-1)^n (x+1)^n = \frac{1}{2+x}$

* ~~Expr~~ Express the following repeated decimals as ratio of two integres :-

$\boxed{1}$ $0.\bar{7} = 0.777777...$

$= 0.7 + 0.07 + 0.007 + 0.0007 ... = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000}$

Loa

$r = \frac{\frac{7}{100}}{\frac{7}{10}} = \frac{1}{10} = \frac{\frac{7}{1000}}{\frac{7}{100}} = \frac{1}{10}$

↳ Geometric series $\Rightarrow r = \frac{1}{10} \in (-1,1) \Rightarrow$ series Conv to

$* \text{sum} = \frac{a}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \boxed{\frac{7}{9}}$ #

$\boxed{2}$ $0.\overline{23} = 0.23232323 ...$

$= \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + ...$

Loa

$r = \frac{\frac{\cancel{23}}{(100)^{\cancel{2}}}}{\frac{\cancel{23}}{\cancel{100}}} = \boxed{\frac{1}{100}}$

↳ Geometric series $\Rightarrow r = \frac{1}{100} \in (-1,1) \Rightarrow$ series Conv to

$* \text{sum} = \frac{a}{1-r} = \frac{\frac{23}{100}}{1-\frac{1}{100}} = \boxed{\frac{23}{99}}$

3 $0.0\overline{5} = 0.\overline{5} \times 10^{-1}$

$= \frac{5}{9} \times 10^{-1} = \frac{5}{9} \times \frac{1}{10} = \boxed{\frac{5}{90}}$

OR $0.0\overline{5} = 0.055555\ldots$

$= \frac{5}{100} + \frac{5}{1000} + \cdots$

a

$r = \frac{\frac{5}{1000}}{\frac{5}{100}} = \boxed{\frac{1}{10}}$

$\ast \text{sum} = \frac{a}{1-r} = \frac{\frac{5}{100}}{1-\frac{1}{10}} = \boxed{\frac{5}{90}}$

= ✓

***Exp:-** Find the sum $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + 3^{\frac{1}{n}}\right)$

$= \sum_{n=0}^{\infty} 2^{\frac{5}{n}} + \sum_{n=0}^{\infty} 3^{\frac{1}{n}}$

$= \underbrace{\left[\overset{a}{5} + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots\right]}_{r = \frac{1}{2} \in (-1,1) \text{ conv.}} + \underbrace{\left[\overset{a}{1} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right]}_{r = \frac{1}{3} \in (-1,1) \text{ conv.}}$

$= \frac{5}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}}$

$= \frac{5}{\frac{1}{2}} + \frac{1}{\frac{2}{3}} = \frac{2}{5} + \frac{3}{2} = \boxed{\frac{23}{2}}$ # ✓

* Ch (10.3) :- Integral Test (I.T)

* This test use to check converge OR diverge of series.

* Consider $\sum_{n=K}^{\infty} a_n$, where

- a_n positive term
- $a_n = f(n)$ is cont., $\oplus$, $\downarrow$ on $[K, \infty)$.

Then $\sum_{n=K}^{\infty} a_n$ and $\int_K^{\infty} f(x)\,dx$, both conv. OR both div.

*Exp :- Does the following series conv./div ?

$\boxed{1}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

* $a_n = \frac{1}{n^2}$ positive term $\forall\, n = 1, 2, 3, \ldots$

* $f(x) = \frac{1}{x^2}$ is $\oplus$, Cont., $\downarrow$ on $[1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \frac{1}{\boxed{2}-1} = \frac{1}{1} = \boxed{1} \quad \text{By Exp*}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by I.T

XX $\boxed{1}$ الـ Conv. Sieries الـ أقول إنه ممنوع ** صوت **

$\boxed{2}$ $\sum_{n=1}^{\infty} \frac{1}{n^3}$, Conv. by (I.T)

* Test :- P-series Test :

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ Conv. if $P>1$

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ div. if $P \leqslant 1$

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, div by (I.T) because this P-series , $P = \frac{1}{2}$

$\boxed{4}$ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

* $a_n = \frac{1}{n^2+1}$ Positive terms $\forall$ $n = 1, 2, 3, \ldots$

* $f(x) = \frac{1}{x^2+1}$ is $\oplus$, Cont, $\downarrow$ on $[1, \infty)$

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1}x \Big|_1^b$$

$$= \lim_{b\to\infty} \left[\tan^{-1}(b) - \tan^{-1}(1) \right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}} \ \#$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Conv. by (I.T)

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5 $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

* $a_n = \frac{1}{2n-1}$ is positive for all $n = 1, 2, 3, \ldots$

* $f(x) = \frac{1}{2x-1}$ is $\downarrow$, $\oplus$, Cont. on $[1, \infty)$.

$\hookrightarrow \int_1^{\infty} \frac{1}{2x-1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{2x-1}\,dx$

$u = 2x - 1$

$du = 2dx$

$\hookrightarrow \frac{du}{2} = dx$

$x = 1 \Rightarrow u = 1$

$x = b \Rightarrow u = 2b - 1$

$= \lim_{b\to\infty} \int_1^{b} \frac{1}{2} \frac{du}{u}$

$= \lim \frac{1}{2} \ln|u| \Big|_1^{2b-1}$

$= \frac{1}{2}\left[\lim_{b\to\infty}\left[\ln(2b-1) - \cancel{\ln|1|}\right]\right]$ zero

$= \frac{1}{2} \lim_{b\to\infty} \ln(2b-1) = \frac{1}{2} \ln\left(\lim_{b\to\infty}(2b-1)\right)$

$= \boxed{\infty}$

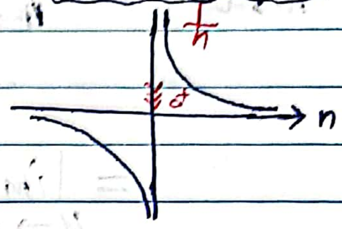
Hence, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ div. by (I.T)

$\boxed{6}$ $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$

$$= \lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$$

$u = \frac{1}{n}$

$n \to \infty \Rightarrow n = 0^+$



$$= \lim_{n\to 0^+} \frac{\sin u}{u} = \boxed{1} \neq 0$$

Hence, $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$ $\underset{\text{div.}}{\downarrow}$ by n^{th} term test.

Q. 26) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

* $a_n = \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ positive terms $\forall\ n = 1, 2, 3, \ldots$

* $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$ is $\oplus$, $\downarrow$, Cont. on $[1, \infty)$

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

$u = \sqrt{x} + 1$

$du = \frac{1}{2\sqrt{x}}$

$2du = \frac{dx}{\sqrt{a}}$

$$= \lim_{b\to\infty} \int_1^{b} \frac{2\,du}{u} = \lim_{b\to\infty} 2 \ln|u| \Big|_1^b$$

$$= 2 \lim \ln|\sqrt{x}+1| \Big|_1^b = 2 \lim \left[\ln(b+1) - \ln(2)\right]$$

$$= \infty - \ln 2 = \boxed{\infty}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ div by (I.T)

Q.34) $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \tan\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{u\to 0^+} \frac{\tan u}{u} \quad \frac{0}{0} \text{ Using L'Hopital Rule.}$$

$$= \lim_{u\to 0^+} \frac{\sec^2 u}{1} = \lim_{u\to 0^+} \frac{1}{\cos^2 u} = \frac{1}{1^2} = \boxed{1} \neq 0$$

Hence, $\sum_{n=1}^{\infty} n \tan\frac{1}{n}$ div by n^{th} term test.

Q. 8) Use (I.T) to check Conv. / Div

$$\sum_{n=2}^{\infty} \frac{\ln n^2}{n}$$

* $a_n = \frac{\ln n^2}{n}$ Positive term.

* $f(x) = \frac{\ln x^2}{x}$ positive on $[2,\infty)$

cont. on $[2,\infty)$

$\downarrow$ because $f'(x) = \frac{x\left(2\frac{1}{x}\right) - \ln x^2}{x^2}$

$$= \frac{2 - \ln x^2}{x^2} \quad = ?? \quad x^2 \to (+) \uparrow$$

$\Rightarrow 2 - \ln x^2 = 0$

$2 = \ln x^2 \Rightarrow 2 = 2\ln x \Rightarrow \ln x = 1 \Rightarrow \boxed{x = e}$

بأخذ e للطرفين

$f(x)$ ونبحث إشارة

f' + + + + + + + + − − − − − − − − −

e $\quad e^2$

≈ 2.718

الازم هوش نفصل لأنه التسلسلة تبدأ من $n=2$ وال e [صغير من e] فيوجد هناك فترة موجبة وأخرى سالبة.

$$\sum_{n=2}^{\infty} \frac{\ln n^2}{n} = \frac{\ln 2^2}{2} + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$$

b_n ⊕, ⊕, cont on $[3,\infty)$.

$$= \ln 2 + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$$

$$* \int_3^{\infty} f(x)\,dx = \int_3^{\infty} \frac{\ln x^2}{x}\,dx = \lim_{b\to\infty} \int_3^b \frac{2\ln x}{x}$$

$\boxed{u = \ln x \quad du = \frac{1}{x}\,dx}$

$$= \lim_{b\to\infty} \int_3^b 2u\,du = \lim_{b\to\infty} u^2 \Big|_3^b$$

$$= \lim_{b\to\infty} (\ln x)^2 \Big|_3^b = \lim \left((\ln(b))^2 - (\ln(3))^2 \right)$$

$$= \boxed{\infty}$$

Hence, $\sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ div by (I.T) / Hence, $\sum_{n=2}^{\infty} \frac{\ln n^2}{n} = \ln 2 + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ div.

*Ch(10.4) :- Comparison test :-

[1] Direct Comparison test :-

* Let $\sum a_n$, $\sum c_n$, $\sum d_n$ be series with non negative terms

* Suppose that $d_n \le a_n \le c_n$ for all $n > N$

الكبير

(a) If $\sum c_n$ Converges, then $\sum a_n$ also Converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

الصغير

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*Exp :- Check for Conv./Div. :-

[1] $\sum_{n=1}^{\infty} \frac{3+\sin n}{n^2} \rightarrow a_n > 0$

Conv. ??

$$\quad \le \frac{3+\sin n}{n^2} \le \frac{4}{n^2}$$

$\frac{4}{n^2}$ Conv. by p-series Test because $p > 1$

Hence, $\sum \frac{3+\sin n}{n^2}$ Conv. by D.C.T

---

[2] $\sum_{n=1}^{\infty} \frac{7n}{5n+1}$

div. by n^{th} term test since $\lim_{n\to\infty} \frac{7n}{5n+1} = \frac{7}{5} \ne 0$

$\boxed{3}$ $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n}$

div. ?? $\qquad\qquad\qquad$ Conv. ??

$$\bigcirc \leq \sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n} \leq \sum_{n=1}^{\infty}\left(\frac{n}{3n}\right)^{n}$$

$$\sum_{n=1}^{\infty}\left(\frac{n}{3n}\right)^{n} = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n} \quad \text{Convergent Geometric Series}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{n} = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \boxed{\frac{1}{2}} \ \# \checkmark$$

Hence, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^{n}$ conv. by D.C.T

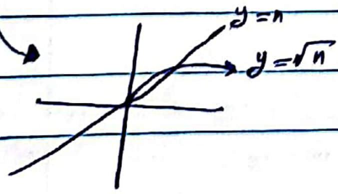
$\boxed{4}$ $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n+\sqrt{n}}$ Use D.C.T.

div. ?? $\qquad\qquad\qquad$ Conv. ??

$$\bigcirc \leq \sum \frac{3}{n+\sqrt{n}} \leq \bigcirc$$

$n = 1, 2, 3, 4, \ldots$

$n > \sqrt{n}$ for large n



$n > \sqrt{n}$

نحصل على صيغة قريبة من صيغة a_n

نجمع للطرفين n

$2n > n+\sqrt{n}$

$$\frac{1}{2n} < \frac{1}{n+\sqrt{n}} \Rightarrow \boxed{\frac{3}{2n} < \frac{3}{n+\sqrt{n}}}$$

$$\sum_{n=1}^{\infty} \frac{3}{2n} < \sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$$

$\frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}$, is div. because Harmonic series
OR P-series, $p=1$

Hence, $\sum \frac{3}{n+\sqrt{n}}$ div by D.C.T.

[2] Limit Comparsion Test:-

$\sum_{n=1}^{\infty} a_n = ??$

* Find b_n s.t $\sum b_n$ is Known // $a_n > 0$, $b_n > 0$

- $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then both $(\sum a_n$ and $\sum b_n)$ are conv
  Lofinite
  OR both $(\sum a_n$ and $\sum b_n)$ are div.

- $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ Conv., then $\sum a_n$ Conv.

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div, then $\sum a_n$ div.

* Expr Check for Conv./Div by L.C.T:-

[1] Q.46) $\sum_{n=1}^{\infty} \tan\frac{1}{n} \longrightarrow a_n$

* $b_n = \frac{1}{n}$ such that $\sum_{n=1}^{\infty} \frac{1}{n}$ div by Harmoine-series

* $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{\sin\frac{1}{n}}{\cos\frac{1}{n}}}{\frac{1}{n}}$

$u = \frac{1}{n}$

$= \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cdot \lim_{n\to\infty} \cos\frac{1}{n}$

$= \lim_{n\to 0^+} \frac{\sin u}{u} \cdot \lim_{n\to 0^+} \cos u$

$= (1)\cdot \cos(0) = (1)\cdot(1) = \boxed{1} > 0$

Hence $\sum \tan\frac{1}{n}$ div by L.C.T

[2] Q.36) $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2\cdot 2^n} \longrightarrow a_n$

* $b_n = \frac{1}{n^2}$ such that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by P-series.

* $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+2^n}{\cancel{n^2}\cdot 2^n}}{\frac{1}{\cancel{n^2}}} = \lim_{n\to\infty} \frac{n+2^n}{2^n}$ $\boxed{\frac{\infty}{\infty}}$

$$\lim_{n\to\infty} \frac{1+2^n \ln 2}{2^n \ln 2} \left[\frac{\infty}{\infty}\right] = \lim \frac{2^n (\ln 2)^2}{2^n (\ln 2)^2} = \boxed{1} > 0.$$

Hence, $\sum \frac{n+2^n}{n^2 \cdot 2^n}$ Conv. by L.C.T

Using D.C.T

$$\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 \cdot 2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{n 2^n} + \frac{1}{n^2} \right)$$

$$\sum \frac{1}{n 2^n} + \frac{1}{n^2} \leq \sum \frac{1}{2^n} + \sum \frac{1}{n^2}$$

Conv. Geometric series

Conv. P-series

Hence, $\sum \frac{n+2^n}{n^2 \cdot 2^n}$ Conv by D.C.T

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{n+2}{n^3+n^2+5} \rightarrow a_n$

$* \; b_n = \frac{1}{n^2}$ such that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Conv. P-series

$* \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+2}{n^3+n^2+5}}{\frac{1}{n^2}} = \lim \frac{n+2}{n^3+n^2+5} \cdot \frac{n^2}{1} = \boxed{1} > 0$

Hence, $\sum \frac{n+2}{n^3+n^2+5}$ Conv. by L.C.T

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* Q.10) $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}} \rightarrow a_n$

$* \; b_n = \frac{1}{\sqrt{n}}$ such that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div. p-series.

$* \lim \frac{a_n}{b_n} = \lim \frac{\sqrt{\frac{n+1}{n^2+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n^2+2}} \cdot \frac{\sqrt{n}}{1}$

$= \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}} = \sqrt{\lim \frac{n^2+n}{n^2+2}} = \sqrt{1} = \boxed{1} >$

Hence, $\sum \sqrt{\frac{n+1}{n^2+2}}$ div. by L.C.T.

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*(Q.22) $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$, Use L.C.T, and D.C.T

*L.C.T:-

$*b_n = \frac{n^1}{n^{2.5}} = \frac{1}{n^{1.5}} = \frac{1}{n^{\frac{3}{2}}} \Rightarrow \sum \frac{1}{n^{\frac{3}{2}}}$ Conv. p-series

$* \lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{n+1}{n} = \boxed{1} > 0$

Hence, $\sum \frac{n+1}{n^2\sqrt{n}}$ Conv. by L.C.T

*D.C.T:-

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} < \sum_{n=1}^{\infty} \frac{n+\boxed{n}}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{2n}{n^2\sqrt{n}}$$

$$= 2\sum \frac{1}{n^{\frac{3}{2}}}$$

conv. p-series

Hence, $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ Conv. by D.C.T

* Ch(10.5) :- The Ratio and Root Tests :-

* Ratio Test :- (R.T) :-

Consider the infinite series $\sum a_n$ with positive terms.

Assume $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, Then

- If $\rho < 1$, then the series Converges.
- If $\rho > 1$, then the series diverges.
- If $\rho = \infty$, then the series diverges.
- If $\rho = 1$, then the test is inconclusive.

*Exp :- Check Conv./Div. :-

$\boxed{1}$ $\sum\limits_{n=1}^{\infty} \frac{n^2}{e^n}$

* $a_n = \frac{n^2}{e^n}$ is positive $\forall$ $n = 1, 2, 3, \ldots$

* $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{\cancel{e^{n+1}}_{e}} \cdot \frac{\cancel{e^n}}{n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{e \cdot n^2} = \left(\frac{1}{e}\right) \cdot \lim \frac{n^2+2n+1}{n^2}$$

$$= \left(\frac{1}{e}\right) \cdot (1) = \boxed{\frac{1}{e}} < 1$$

Hence, $\sum \frac{n^2}{e^n}$ Conv. by (R.T)

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Q.6) $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

$a_n > 0$ this positive $\forall\, n = 1, 2, 3, \ldots$

* $a_n = \frac{3^{n+2}}{\ln n}$

* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{3^{n+3}}\, 3^1}{\ln(n+1)} \cdot \frac{\ln n}{\cancel{3^{n+2}}}$

$= 3 \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}$ $\boxed{\frac{\infty}{\infty}}$ L'Hopital Rule.

$= 3 \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 3 \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$

$= (3)\cdot(1) = \boxed{3} > 1$

Hence, $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$ div. by (R.T)

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ Div. since it's the harmonic series.

* $a_n = \frac{1}{n}$ is positive $\forall n = 1, 2, 3, 4, \dots$

* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \boxed{1} = \boxed{1}$

↳ * R.T fails to make decision, so we try to apply another test.

$\boxed{4}$ $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

* $a_n = \frac{n!}{e^n}$ is positive $\forall n = 1, 2, 3, \dots$

* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \lim_{n\to\infty} \frac{n+1}{e} = \boxed{\infty} > 1$

Hence, $\sum \frac{n!}{e^n}$ div. by (R.T)

$\boxed{5}$ (Q.20) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

* $a_n = \frac{n!}{10^n}$ is positive $\forall n = 1, 2, 3, \dots$

* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{1}{10} \lim_{n\to\infty} (n+1)$

$= \boxed{\infty} > 1$

Hence, $\sum \frac{n!}{10^n}$ div by (R.T)

CamScanner

* **Root Test:-**

Consider the infinite series $\sum a_n$ with $a_n \geq 0$ for $n \geq N$

Assume $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$. Then,

- If $\rho < 1$, then the series converges.
- If $\rho > 1$, then the series diverges.
- If $\rho = \infty$, then the series diverges.
- If $\rho = 1$, then the test is inconclusive.

يتم استخدام هذا ال Ratio Test لل Vectorial بشكل كبير

* **Exp:-** Apply the root test to:-

$\boxed{1}$ $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

* $a_n = \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

* $\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 < 1$

Hence, $\sum \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$ conv. by Root test.

$\boxed{2}$ $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

$* \; a_n = \frac{(\ln n)^n}{n^n} = \left(\frac{\ln n}{n}\right)^n$

$* \lim_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{\ln n}{n}\right) = \boxed{0} < 1$

Hence, $\sum \frac{(\ln n)^n}{n^n}$ Conv by the Root terst.

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

$* \; a_n = \frac{3^n}{n^3} > 0 \quad \forall n = 1, 2, 3, \dots$

$* \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^3}} = \lim_{n\to\infty} \frac{3}{\sqrt[n]{n^3}} = 3 \lim_{n\to\infty} \frac{1}{(n^3)^{\frac{1}{n}}}$

$= 3 \lim_{n\to\infty} \frac{1}{n^{\frac{3}{n}}} = \frac{3}{\lim_{n\to\infty} (n^{\frac{1}{n}})^3} = \frac{3}{\lim_{n\to\infty} (\sqrt[n]{n})^3} = \frac{3}{(1)^3}$

$= \boxed{3} > 1$

Hence, $\sum \frac{3^n}{n^3}$ div. by Root test.

Q. 15) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

$* \; a_n = \left(1-\frac{1}{n}\right)^{n^2}$

$* \lim_{n\to\infty} \sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left[\left(1-\frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n}$

$= e^{-1} = \boxed{\frac{1}{e}} < 1$

Hence, $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$ conv. by Root Test.

Q. 30) $\sum \left(\frac{1}{n}-\frac{1}{n^2}\right)^{n}$

** تم حل السؤال مسبقاً **

* Expl: Recursive terms

$a_1 = 2$, $a_{n+1} = \frac{2}{n} a_n$ $\frac{a_{n+1}}{a_n} = \frac{2}{n}$

Does $\sum_{n=1}^{\infty} a_n$ conv. ??

Apply Ratio Test $\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n\to\infty} \frac{2}{n} = \boxed{0} < \boxed{1}$

Yes, it does conv. by (R.T)

* Ch (10.6) :- Alternating series :-

- The alternating series has the form

$$\sum_{n=1}^{\infty} a_n := \sum_{n=1}^{\infty} (-1)^{n+1} u_n = U_1 - U_2 + U_3 - U_4 + \cdots$$

- **Th :- Alternating series Test :- (A.S.T) :-**

$\sum (-1)^{n+1} u_n$ Conv. if :-

$\boxed{1}$ $u_n > 0 \ \forall n \Rightarrow u_n = |a_n|$

$\boxed{2}$ u_n deeresing ($\downarrow$) for larg n. $\Rightarrow u_{n+1} \leq u_n$

$\boxed{3}$ $\lim_{n\to\infty} u_n = 0 \Rightarrow$ (If not $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} u_n$ div by n^{th} term test)

* Exp :- Check Conv. / div. for :-

① $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \Rightarrow$ (Alternating Harmonic Series)

* Apply A.S.T :-

$a_n = (-1)^{n+1} \cdot \frac{1}{n} \Rightarrow u_n = |a_n| = \frac{1}{n}$

$u_n > 0$, $u_n \downarrow$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ Conv. by A.S.T

$$-\frac{2}{10} + \left(\frac{2}{10}\right)^2 - \left(\frac{2}{10}\right)^3 + \cdots \Rightarrow \text{Conv. to } \frac{-\frac{2}{10}}{1-\left(-\frac{2}{10}\right)} = \boxed{\frac{-1}{6}}$$

$\boxed{2}$ $\sum_{n=1}^{\infty} (-1)^n (0.2)^n$ (Alternating Geometric Series)

* Apply (A.S.T)

$a_n = (-1)^n (0.2)^n \Rightarrow u_n = (0.2)^n = \left(\frac{1}{5}\right)^n = \frac{1}{5^n}$

$u_n > 0$, $u_n \downarrow$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{5^n} = \lim_{n\to\infty} (0.2)^n = 0 \rightarrow$ Th.5 in [o.]

Hence, $\sum_{n=1}^{\infty} (-1)^n (0.2)^n$ Conv. by (A.S.T)

$\boxed{3}$ $\sum_{n=1}^{\infty} (-1)^n n$

* Apply (A.S.T)

$u_n = n$

$u_n > 0$, u_n is not decreasing, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} n = \infty \neq 0$

$\hookrightarrow \sum (-1)^n n$ div by n^{th} term test

$\boxed{4}$ $\sum_{n=3}^{\infty} (-1)^n \frac{2n}{3n-4}$

$u_n = \frac{2n}{3n-4}$

$\lim_{n\to\infty} \frac{2n}{3n-4} = \frac{2}{3} \neq 0 \Rightarrow$ Hence, $\sum_{n=3}^{\infty} (-1)^n \frac{2n}{3n-4}$ div by n^{th} term test.

* ملاحظات على الـ Test *

* اذا تحققت كافة الشروط ⟸ Conv.

* اذا تحقق الشرطان 1 , 2 ولم يتحقق الأخير ⟸ div.

* اذا تحقق 1 , 3 ولم يتحقق الـ 2 ⟸ لا نستطيع الحكم.

• **Def**: (Conv. Abs.)

$\sum a_n$ Conv. Abs if $\sum |a_n|$ Conv.

* **Exp**: Check if $\sum a_n$ Conv. Abs. ?

$\boxed{1}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} \to a_n$

↳ $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ ⇒ Conv. by P-series

Hence, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ Conv. Abs.

---

* **Th**: If $\sum |a_n|$ Conv. ⇒ $\sum a_n$ Conv.

↳ Means If $\sum a_n$ Conv. Abs "$\sum |a_n|$ Conv." then $\sum a_n$ Conv.

Important:

if $\sum a_n$ Conv. $\not\Rightarrow \sum a_n$ conv. Abs

ليس بالضرورة

* **Def:** (Conv. Conditionally):

$\sum a_n$ Conv. Cond. If it Conv. by (A.S.T) but not Abs.

* **Remark:**

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p} = \begin{cases} \text{Conv. Abs.} & \text{if } p > 1 \\ \text{conv. Condi} & \text{if } 0 < p \leq 1 \\ \text{div} & \text{if } p \leq 0 \end{cases}$$

$\hookrightarrow$ by n^{th} term test

* **Exp:**

$\boxed{1}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$ Conv. Abs $\Longrightarrow$ Conv.

Since, $\sum |a_n| = \sum \frac{1}{n^3}$ Conv. by p-series

$\boxed{2}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{\frac{2}{3}}}$ Conv. Conditionally ($0 < p = \frac{2}{3} \leq 1$)

$\hookrightarrow$ Conv. but not Abs. $\rightarrow \sum |a_n| = \sum \frac{1}{n^{\frac{2}{3}}}$ by p-series div, $p = \frac{2}{3} < 1$

$\hookrightarrow$ by (A.S.T)

$\boxed{3}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{-2}} = \sum_{n=1}^{\infty} (-1)^{n+1} n^2$

div by nth term test

→ $u_n = n^2$, $\lim_{n\to\infty} n^2 = \infty \neq 0 \Rightarrow \underline{\underline{\text{div}}}$

* Th :- (Alternating Estimation Th) :-

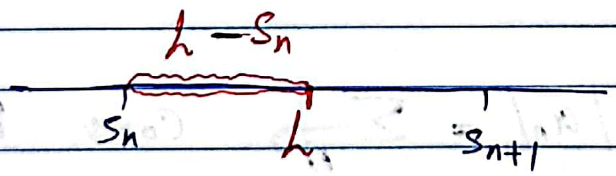
- Assume $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n = U_1 - U_2 + U_3 + \dots = L$
- If we approximate L by $S_n = U_1 - U_2 + U_3 + \dots + (-1)^{n+1} U_n$

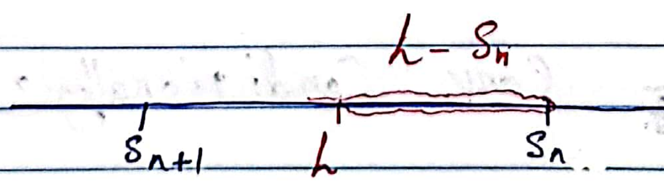
then $\boxed{1}$ the remainder $= \boxed{L - S_n} \Rightarrow$ has same sign as a_{n+1}

$\boxed{2}$ the error $|L - S_n| < u_{n+1} = |a_{n+1}|$

$\boxed{3}$ $\min\{S_n, S_{n+1}\} < L < \max\{S_n, S_{n+1}\}$

توضيح





+ EXP:- $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2}{3}\right)^n = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 + \cdots$

$$= \frac{\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{4}{10} = 0.4 = L$$

if we approximate $L = 0.4$ by $S_3 = \left(\frac{2}{3}\right) - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$

ملاحظة أن $n \geq 3$ $\qquad \approx \frac{14}{27} \approx 0.519$

[1] Remainder $= L - S_n = 0.4 - 0.519 = \boxed{-0.119}$

إشارة نفس إشارة a_{n+1}

$\hookrightarrow a_{3+1} = a_4 = -\left(\frac{2}{3}\right)^4$

إشارة السالب

[2] Error $= |L - S_n| = |L - S_3| = |-0.119| = \boxed{0.119}$ #✓

$E = 0.119 < U_n = |a_4| = \left(\frac{2}{3}\right)^4 = 0.198$

[3] $S_4 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^4 \approx 0.321$

$\min\{S_3, S_4\} < L < \max\{S_3, S_4\}$

$\min\{0.519, 0.321\} < 0.4 < \max\{0.519, 0.321\}$

$0.321 < 0.4 < 0.519$ ✓

* Exp:- Approximate sum of $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$

With error of magnitude less than 5×10^{-6}.

- We use S_n to approx. the sum

So we need to find $n \Rightarrow \frac{1}{(2n)!} < 5\times10^{-6}$

$(2n)! > \frac{1}{5\times10^{-6}}$

$\hookrightarrow (2n)! > \frac{10^6}{5} = 200{,}000$

$\hookrightarrow (2n)! > 200{,}000 \Rightarrow n \geq 5$

- $S_5 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54$

- Error $= |L - S_n| = |L - S_5| < u_6 = \frac{1}{(10)!} = 0.275\times10^{-6}$

Q.14) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1} \rightarrow u_n = |a_n|$

$$u_n = \frac{3\sqrt{n+1}}{\sqrt{n}+1}$$

نفحص الشرط ③ $\lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = \frac{3}{1} = \boxed{3} \neq 0$

So, the alternating series div. by n^{th} term test.

Q.19) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$

① $u_n > 0$, ② $u_n \downarrow$

③ $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^3+1} = 0$ ✓

So, this series Conv. by (A.S.T)

To check if it conv. Abs. $\Rightarrow$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$n^3+1 > n^3$

$\frac{1}{n^3+1} < \frac{1}{n^3} \Rightarrow \frac{n}{n^3+1} < \frac{n}{n^3} \Rightarrow \frac{n}{n^3+1} < \frac{1}{n^2}$

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$\hookrightarrow \sum \frac{1}{n^2}$ conv.

$\hookrightarrow \sum |a_n|$ conv. by (D.C.T)

$\hookrightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$ Conv. Abs.

Q 29) $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n^2+1}$

$$U_n = \frac{\tan^{-1} n}{n^2+1}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$$ → cont, ↓, ⊕

We use (I.T)

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1} dx = \lim_{b \to \infty} \int_1^{b} \frac{\tan^{-1} x}{x^2+1} dx$$

$u = \tan^{-1} x$

$du = \frac{dx}{x^2+1}$

$$= \lim_{b \to \infty} \int_{\frac{\pi}{4}}^{\tan^{-1} b} u \, du$$

$x=1 \Rightarrow \frac{\pi}{4}$

$x=b \Rightarrow \tan^{-1} b$

So, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tan^{-1} n}{n^2+1}$ Conv. Abs

$$= \lim_{b \to \infty} \frac{u^2}{2} \Big|_{\frac{\pi}{4}}^{\tan^{-1} b}$$

$$= \frac{1}{2} \lim_{b \to \infty} u^2 \Big|_{\frac{\pi}{4}}^{\tan^{-1} b} = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \boxed{\frac{3\pi^2}{32}}$$ Conv.

Power series

Conv — div

Abs. — Condi. → الفحص على حدود الفترات

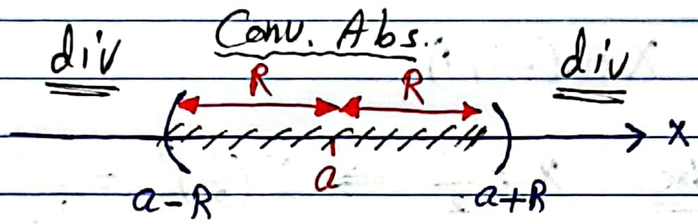
*Ch (10.7):- Power Series:-

- Power series are infinit sum of Poly's

كثيرات حدود

- $\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$

* Such that:- a_1 : Coefficients.
a : Center



* Such that:- R: Radius of Convergance.
IC: Interval of Convergance.

قديش ممكن أبعد عن a لنقطة ثانية عندها converges

** $IC = |x-a| < R$

$-R < x-a < R$

$a-R < x < a+R$

* Note:-
To find (R) and (IC)
We apply (Ratio test)

---

* Exp:-

$\boxed{1}$ $\sum_{n=0}^{\infty} x^n$

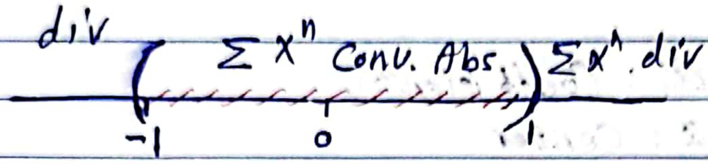
Center : (a) عند إيجاد → $\sum a_n (x-a)^n$

تقارن بالشكل العام للـ Power series

↳ a : center = 0 (Zero)

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \Rightarrow \text{Geometric since } \boxed{r = x}$$

If $|x| < 1 \Rightarrow \sum x^n$ Converges to $\frac{1}{1-x}$ by Geometric series



$x \in (-1, 1)$

*Take: $x = \frac{1}{2} \Rightarrow \sum \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$

$x = 2 \Rightarrow \sum 2^n$ div by n^{th} term test.

since $\lim_{n\to\infty} 2^n = \infty$

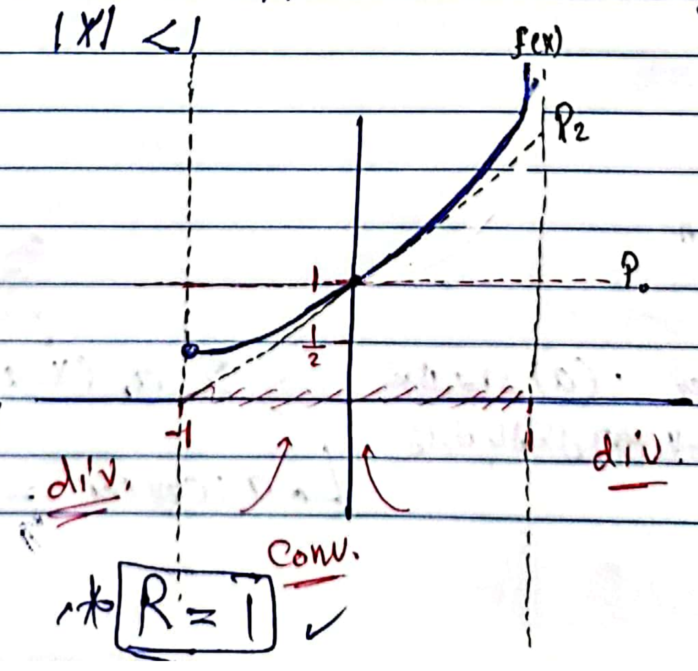
$f(x) = \frac{1}{1-x} \Rightarrow$ We can approximate $f(x)$ by $|x| < 1$

$P_0 = 1$

$P_1 = 1 + x$

$P_2 = 1 + x + x^2$

$P_3 = 1 + x + x^2 + x^3$



* $IC = (-1, 1)$ Conv. Abs.

* $\boxed{R = 1}$ ✓

*EXP: Find R and IC

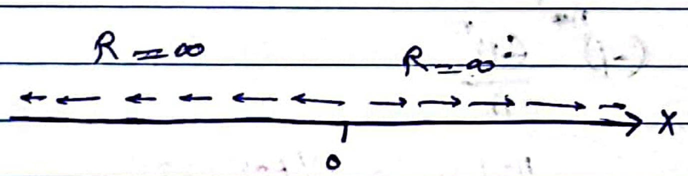
$\boxed{1}$ $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ a_n $\to$ a: center = zero

To find R, IC We Apply R.T.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{\,x}}{\cancel{(n+1)!}_{\,(n+1)}} \cdot \frac{\cancel{n!}}{\cancel{x^n}}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$= |x| (0) = \boxed{0} < 1$$



*$R = \infty$

*$IC = \mathbb{R}\ (-\infty, \infty)$ } $\sum \frac{(-1)^n x^n}{n!}$ Conv. Abs. for every x.

$\boxed{2}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\to a_n$ $\Rightarrow$ a: center = zero

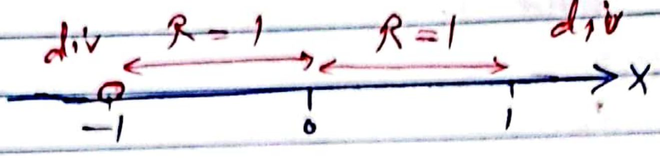
Apply R.T

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{\,x}}{n+1} \cdot \frac{n}{\cancel{x^n}}\right| = |x| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x|.(1) = |x| < 1$$

$-1 < x < 1$ Conv. Abs

Conv. Abs.



* To Check conv. condit

└ Check end Point.

* $X = -1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$

$= \sum (-1)^{2n} (-1)^{-1} \frac{1}{n}$

$= \sum (1) \left(\frac{1}{-1}\right) \frac{1}{n} = -\sum \frac{1}{n}$ div

because this is series is harmonic

* $X = 1 \Rightarrow \sum (-1)^{n-1} \frac{(-1)^n}{n}$

$= \sum \frac{(-1)^{n-1}}{n} \longrightarrow$ Alternating harmonic series

└ conv.

└ $X = 1 \Rightarrow \sum \frac{(-1)^{n-1}}{n}$ is not conv. Abs

since, $\sum |a_n| = \sum \frac{1}{n}$ div

* $R = 1$

* $IC = [-1, 1]$.

* at $X = 1 \Rightarrow$ conv. Conditionaly.

* Conv. Abs only $(-1, 1)$

Q. 24) $\sum_{n=1}^{\infty} \underbrace{(\ln n)\, x^n}_{\to a_n}$ → a : center = zero

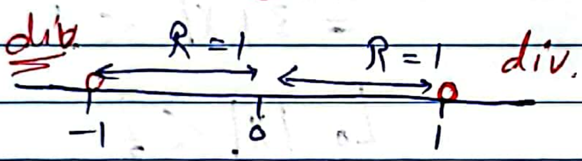
Apply R.T

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\ln(n+1)\; x^{n+1}}{\ln n \quad x^n} \right| = |x| \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \quad \frac{\infty}{\infty}$$

$$= |x| \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|(1)$$

$$= |x| < 1$$

Conv. Abs



$-1 < x < 1$

Conv. Abs.

↳ $(-1, 1)$ conv. Abs

* We check end point :-

$$x = 1 \Rightarrow \sum_{n=1}^{\infty} \ln n \; x^n = \sum_{n=1}^{\infty} (\ln n)1^n = \sum_{n=1}^{\infty} \ln n$$

div by n^{th} term test

$$x = -1 \Rightarrow \sum_{n=1}^{\infty} \ln n \, (-1)^n$$

Alternating series

(3) [illegible]

$$\lim_{n\to\infty} \ln n = \infty \neq 0 \Rightarrow \text{div}$$

$\nexists$ x such that $\sum \ln n \, (x)^n$ conv. conditional.

* $R = 1$

* IC $= (-1, 1)$ conv. Abs.

Q. 41) Find IC, R, Conv. Abs^y. conv. Conditionaly :

$$\sum_{n=0}^{\infty} 3^n x^n \to a_n \qquad a : \text{Center} = \text{Zero}$$

Apply R.T.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{3^{n+1}} \cdot \cancel{x^{n+1}}}{\cancel{3^n} \cdot \cancel{x^n}}\right| = |x| \lim_{n\to\infty} 3$$

$$= 3|x| < 1$$

$$\hookrightarrow |x| < \frac{1}{3}$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

Conv. Abs

$$\left(-\frac{1}{3}, \frac{1}{3}\right)$$

* We Check end points:

$$x = \frac{-1}{3} \Rightarrow \sum 3^n \left(-\frac{1}{3}\right)^n = \sum \cancel{3^n} \cancel{\left(\frac{1}{3}\right)^n} (-1)^n = \sum (-1)^n$$

div by n^{th} term test.

$$x = \frac{1}{3} \Rightarrow \sum 3^n \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} 1 \quad \text{div by } n^{th} \text{ term test.}$$

Hence $\sum 3^n x^n$ conv. Abs. $\forall\, x \in \left(\frac{-1}{3}, \frac{1}{3}\right)$ IC

$$\boxed{R = \frac{1}{3}}$$ #

$$\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n = 1 + 3x + (3x)^2 + (3x)^3 + \cdots$$

$|x| < \frac{1}{3}$

Conv. geometric series

$$= \frac{1}{1 - (3x)^{r}}$$

$|r| < 1$

$|3x| < 1$

$-1 < 3x < 1$

$-\frac{1}{3} < x < \frac{1}{3}$

$|x| < \frac{1}{3}$

*Exp: $\sum_{n=1}^{\infty} \underbrace{\frac{(x-1)^n}{n^3\, 3^n}}_{a_n}$

a: Center $= \boxed{1}$ ✓

بالمقارنة بالصورة العامة للـ power series

Apply R.T.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x-1)^{n+1}}^{(x-1)}}{(n+1)^3 \, \cancel{3^{n+1}}_{3}} \cdot \frac{n^3 \cdot \cancel{3^n}}{\cancel{(x-1)^n}} \right|$$

$$= \frac{1}{3} \cdot |x-1| \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \frac{1}{3}|x-1| < 1$$

$|x-1| < 3$

$-3 < x-1 < 3$

$-2 < x < 4 \Rightarrow (-2, 4)$

Conv. Abs.

div   R = 3   Conv. Abs.   R = 3   div

-2   1   4

$\boxed{R = 3}$ # ✓

* We check end points:

$X = -2 \Rightarrow \sum \frac{(-2)^n}{n^3 \, 3^n}$ — $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ Alternating p-series with $p = 3$

↳ converges by AST

$$\boxed{IC = [-2, 4]}$$ #✓

$X = 4 \Rightarrow \sum \frac{\cancel{3^4}}{n^3 \, \cancel{3^4}} = \sum \frac{1}{n^3}$ → p-series Conv.

↳ The power series conv. $\forall$ $[-2, 4]$

* At $X = -2$, $X = 4$ ⇒ Does the serise conv. Abs?

if — No: $X = -2$, $X = 4$ Conv. conditialy

if — Yes: $X = -2$, $X = 4$ Conv. Abs.

$\nexists$ no points where the power series conv. conditialy.

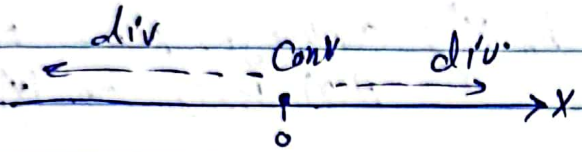
Hence, the power series conv. Abs $\forall x \in [-2, 4]$.

#✓

*Exp:- $\sum_{n=0}^{\infty} n!\, x^n \rightarrow a_n$ $\quad$ a : center = zero

Apply R.T

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}^{\,(n+1)}\ \cancel{x^{n+1}}^{\,x}}{\cancel{n!}\ \cancel{x^n}}\right| = \lim_{n\to\infty}|x(n+1)|$$



$$= |x| \lim_{n\to\infty} n+1$$

$$= \infty > 1$$

This infinit series diverges for every x except $x=0$

*Since, when $x=0$

$$\Rightarrow \sum_{n=0}^{\infty} n!\, x^n = \sum_{n=0}^{\infty} 0 = 0+0+0 - - - - +0 = \boxed{0}$$

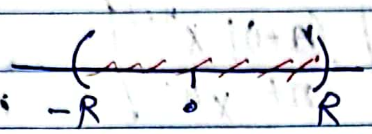
conv.

$$\boxed{R=0}$$

$a=0$ $\quad$ $a=0$

***Th:-** Assume $\sum_{n=0}^{\infty} a_n x^n = A(x)$ and $\sum_{n=0}^{\infty} b_n x^n = B(x)$

Converges Abs. on $|x| < R$



Then, $\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n \cdot x^n\right)$ Converges Abs. to $A(x) \cdot B(x)$ on $|x| < R$

***Th:-** If $\sum_{n=0}^{\infty} a_n x^n$ conv. Abs. on $|x| < R$

then, $\sum a_n (f(x))^n$ conv. Abs on $|f(x)| < R$

for Any cont. function.

***Exp:** $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

$$= \frac{1}{1-x} \quad \text{if } |x| < 1$$

This means $\sum_{n=0}^{\infty} x^n$ Conv. Abs. to $\frac{1}{1-x}$

on $|x| < 1$

$$\sum_{n=0}^{\infty} (4x^2)^n = 1 + 4x^2 + (4x^2)^2 + (4x^2)^3 + \dots$$

$$= \frac{1}{1-4x^2} \quad \text{if } |4x^2| < 1$$

$$4|x^2| < 1$$

$$|x^2| < \frac{1}{4}$$

$$\sqrt{x^2} < \sqrt{\frac{1}{4}}$$

$$|x| < \frac{1}{2}$$

$$-\frac{1}{2} < x < \frac{1}{2} \qquad f = 4x^2 \text{ Conc.}$$

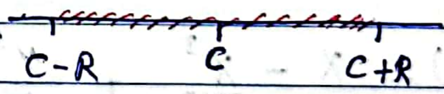
***Th:- Term by Term Differentiation:-**

If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$

Conv. Abs. on $|x-c| < R$

$$\to -R < x-c < R$$

$$c-R < x < R+c$$



If f has all derivatives on $|x-c| < R$, then

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n (x-c)^{n-1} = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n (x-c)^{n-2} = 0 + 0 + 2a_2 + 6a_3(x-c) + \dots$$

* Th :- Term by Term Integration Th :-

Assume $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ Conv. Abs. on $|x-c| < R$

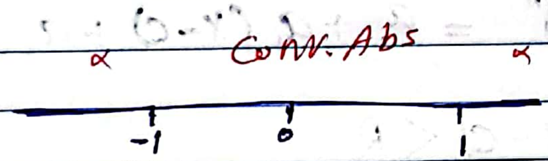
$$= a_0 + a_1 (x-c) + a_2 (x-c)^2 + a_3 (x-c)^3 + \cdots$$

Then, $\int f(x)\,dx = \sum a_n \frac{(x-c)^{n+1}}{n+1} + C$ on $|x-c| < R$

* Exp :- Identify this function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $|x| \leq 1$

$\boxed{a}$ $\sin^{-1} x$, $\boxed{b}$ $\cos^{-1} x$, $\boxed{c}$ $\tan^{-1} x$, $\boxed{d}$ $\sec^{-1} x$

* Compare with $\sum_{n=0}^{\infty} a_n (x-a)^n \Rightarrow a = 0$



$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots$ on $|x| < 1$

Geometric series

$r = |x^2| = x^2 < 1$

$= \frac{1}{1 - \boxed{-x^2}} = \frac{1}{1+x^2} \Rightarrow \int f'(x)\,dx = \int \frac{1}{1+x^2}\,dx$

$f(x) = \tan^{-1} x + C$

$f(0) = 0$

$f(x) = \tan^{-1} x$

Approximation:

$P_0(x) = 1$

$P_2(x) = 1 - \frac{x^2}{2!}$

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

كل ما نزلنا وزدنا الدرجة كل ما كان التقريب أحسن وأحسن للاقتران الأصلي

$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} - \dots + \frac{(-1)x^{2n}}{(2n)!}$. $\cos x$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n!)}$

$\lim_{n \to \infty} P_n(x) = \cos x$

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[2] $f(x) = \sin x$

$f(x) = \sin x \longleftrightarrow f'(x) = \cos x \longrightarrow f''(x) = -\sin x$

$f(0) = 0$ $\quad f'(0) = 1$ $\quad f''(0) = 0$

$f'''(x) = -\cos x \longrightarrow f^{(4)}(x) = \sin x \longrightarrow f^{(5)}(x) = \cos x$

$f'''(0) = -1$ $\quad f^{(4)}(0) = 0$ $\quad f^{(5)}(0) = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

$\boxed{3}$ $f(x) = e^x$

$f(x) = e^x \longrightarrow f'(x) = e^x \longrightarrow f''(x) = e^x \; ----$

$f(0) = 1 \qquad f'(0) = 1 \qquad f''(0) = 1 \; ----$

$$e^x = \sum \frac{f^n(0)}{n!} x^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\left.\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}\right\}$$

*Exp: Find Taylor series of $f(x) = 2^x$ at $x=1$

$\hookrightarrow a=1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \dots$$

$f(x) = 2^x \longrightarrow f'(x) = 2^x(\ln 2) \longrightarrow f''(x) = 2^x(\ln 2)^2$

$f(1) = 2$ $\quad f'(1) = 2\ln 2$ $\quad f''(1)\ 2(\ln 2)^2$

$f'''(x) = 2^x(\ln 2)^3$

$f'''(1)\ 2(\ln 2)^3$

$\vdots$

$f^{(n)} = 2^x(\ln 2)^n$

$$2^x = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \dots$$

$$2^x = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(x-1)^n$$

*Exp: Find MS for $\cosh x$.

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[e^x + e^{-x}\right]$$

$$= \frac{1}{2}\left[\left(1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} \dots\right) + \left(1 + \cancel{(-x)} + \frac{(-x)^2}{2!} + \cancel{\frac{(-x)^3}{3!}} + \dots\right)\right]$$

$$= \frac{1}{2}\left[2 + 2\frac{x^2}{2!} + \frac{2x^4}{4!} + 2\frac{x^6}{6!} \dots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

*Exp:- Find the first 4 nonzero terms in the Ms of

1. $\frac{1}{3}(2x + x\cos x)$

$$= \frac{2}{3}x + \frac{x}{3}\cos x$$

$$= \frac{2}{3}x + \frac{x}{3}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right]$$

$$= \frac{2}{3}x + \frac{x}{3} - \frac{x^3}{3(2!)} + \frac{x^5}{3(2!)} - \frac{x^7}{3(6!)} + \dots$$

$$= x - \dots$$

*The 1^{st} 4 nonzero terms :- x, $\frac{-x^3}{3(2!)}$, $\frac{x^5}{3(2!)}$, $\frac{-x^7}{3(6!)}$

2. $e^x \sin x$

$$= \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right] \cdot \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right]$$

$$= x + x^2 + \frac{x^3}{2!} + \cancel{\frac{x^4}{3!}} + \dots - \frac{x^3}{3!} - \cancel{\frac{x^4}{3!}} - \frac{x^5}{3!(2!)} - \frac{x^6}{(3!)(3!)} + \dots$$

$$+ \frac{x^5}{5!} + \frac{x^6}{5!} + \dots$$

$$= \underbrace{x}_{\text{الحد ١}} + \underbrace{x^2}_{\text{الحد ٢}} + \underbrace{\left[\frac{x^3}{2!} - \frac{x^3}{3!}\right]}_{\text{الحد ٣}} + \underbrace{\left(\frac{x^5}{4!} - \frac{x^5}{3!(2!)} + \frac{x^5}{5!}\right)}_{\text{الحد ٤}}$$

Ch(10.8):- Taylor and Maclaurin Series:-

***Def:-** Let f be a smooth function (All derivatives exist) on an Interval that contains the interior point a, Then

- The Taylor series generated by f at $x=a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots$$

$\longrightarrow a=0$

- The Maclaurin series generated by f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

***Exp:-** Find Maclaurin Series for

$\boxed{1}$ $f(x) = \cos x$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \ldots$$

$f(x) = \cos x \longrightarrow f'(x) = -\sin x \longrightarrow f''(x) = -\cos x$

$\hookrightarrow f(0) = 1$ $\quad$ $\hookrightarrow f'(0) = 0$ $\quad$ $f''(0) = -1$

$f'''(x) = \sin x \longrightarrow f^{(4)}(x) = \cos x \longrightarrow f^{(5)}(x) = -\sin x$

$f'''(0) = 0$ $\quad$ $f^{(4)}(0) = 1$ $\quad$ $f^{(5)}(0) =$

$$= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Approximation:

$P_0(x) = 1$

$P_2(x) = 1 - \frac{x^2}{2!}$

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

كل ما نزلنا وزدنا الدرجة كل ما كان التقريب أحسن وأحسن للاقتران الأصلي $\cos x$

$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \cdots + \frac{(-1)x^{2n}}{(2n)!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n!)}$

$\lim_{n\to\infty} P_n(x) = \cos x$

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$\boxed{2}$ $f(x) = \sin x$

$f(x) = \sin x \to f'(x) = \cos x \to f''(x) = -\sin x$

$f(0) = 0$ $\quad f'(0) = 1$ $\quad f''(0) = 0$

$f'''(x) = -\cos x \to f^{(4)}(x) = \sin x \to f^{(5)}(x) = \cos x$

$f'''(0) = -1$ $\quad f^{(4)}(0) = 0$ $\quad f^{(5)}(0) = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

$\boxed{3}$ $f(x) = e^x$

$f(x) = e^x \longrightarrow f'(x) = e^x \longrightarrow f''(x) = e^x$ ----

$f(0) = 1$; $f'(0) = 1$; $f''(0) = 1$ ----

$$e^x = \sum \frac{f^{n}(0)}{n!} x^n = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\left.\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}\right\}$$

مهم

***Exp:** Find Taylor series of $f(x) = 2^x$ at $x = 1$

$\hookrightarrow a = 1$

$$\sum_{n=0}^{\infty} \frac{f^{n}(1)}{n!}(x-1)^n = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \dots$$

$f(x) = 2^x \longrightarrow f'(x) = 2^x(\ln 2) \longrightarrow f''(x) = 2^x(\ln 2)^2$

$f(1) = 2 \qquad f'(1) = 2\ln 2 \qquad f''(1) = 2(\ln 2)^2$

$f'''(x) = 2^x(\ln 2)^3$

$f'''(1) = 2(\ln 2)^3$

$\vdots$

$f^{(n)} = 2^x(\ln 2)^n$

$$2^x = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \dots$$

$$2^x = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(x-1)^n$$

***Exp:** Find MS for $\cosh x$.

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[e^x + e^{-x}\right]$$

$$= \frac{1}{2}\left[\left(1 + \cancel{x} + \overset{\frac{x^2}{2!}}{\cancel{\frac{x^3}{3!}}} + \frac{x^4}{4!} \dots\right) + \left(1 + \cancel{(-x)} + \cancel{\frac{(-x)^3}{2!}} + \dots\right)\right]$$

$$= \frac{1}{2}\left[2 + 2\frac{x^2}{2!} + \frac{2x^4}{4!} + 2\frac{x^6}{6!} \dots\right]$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

* Exp :- Find the first 4 nonzero terms in the Ms of

① $\frac{1}{3}\left(2X + X\cos X\right)$

$= \frac{2}{3}X + \frac{X}{3}\cos X$

$= \frac{2}{3}X + \frac{X}{3}\left[1 - \frac{X^2}{2!} + \frac{X^4}{4!} - \frac{X^6}{6!} + \cdots\right]$

$= \frac{2}{3}X + \frac{X}{3} - \frac{X^3}{3(2!)} + \frac{X^5}{3(2!)} - \frac{X^7}{3(6!)} + \cdots$

$= X - \cdots$

* The 1st 4 nonzero terms :- X , $\frac{-X^3}{3(2!)}$, $\frac{X^5}{3(2!)}$, $\frac{-X^7}{3(6!)}$

② $e^X \sin X$

$= \left[1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots\right] \cdot \left[X - \frac{X^3}{3!} + \frac{X^5}{5!} + \cdots\right]$

$= X + X^2 + \frac{X^3}{2!} + \cancel{\frac{X^4}{3!}} + \cdots - \frac{X^3}{3!} - \cancel{\frac{X^4}{3!}} - \frac{X^5}{3!(2!)} - \frac{X^6}{(3!)(3!)} + \cdots$

$+ \frac{X^5}{5!} + \frac{X^6}{5!} + \cdots$

$= \underset{\text{الحد ①}}{X} + \underset{\text{الحد ②}}{X^2} + \underset{\text{الحد ③}}{\left[\frac{X^3}{2!} - \frac{X^3}{3!}\right]} + \underset{\text{الحد ④}}{\left(\frac{X^5}{4!} - \frac{X^5}{3!(2!)} + \frac{X^5}{5!}\right)}$

*Ch(10.9):- Convergence of Taylor series:-

* f has all derivatives on $[a,b]$, $c \in (a,b)$

↳ Taylor series of f about $x = c$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

* Maclurine series = Taylor series at $c = 0$

• Maclurine series for:-

$\boxed{1}$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\boxed{2}$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$\boxed{3}$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

* Q.22) Find Maclorine series of $f(x) = \frac{2}{(1-x)^3}$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f = 2(1-x)^{-3} \longrightarrow f(0) = 2$

$f'(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4} \longrightarrow f'(0) = 6$

$f''(x) = 24(1-x)^{-5}(-1) = 24(1-x)^{-5} \longrightarrow f''(0) = 24$

$f'''(x) = -120(1-x)^{-6}(-1) = 120(1-x)^{-6} \longrightarrow f'''(0) = 120$

$$= 2+6x + \frac{24}{2!}x^2 + \frac{120}{3!}x^3 + \dots$$

OR

$$f(x) = \frac{2}{(1-x)^3} = \left(\frac{1}{1-x}\right)'' \longrightarrow \text{Geometric series}$$

a, r

$$= \left(1 + x + x^2 + x^3 + x^4 + x^5 \dots\right)''$$

$$= \left(0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots\right)'$$

$$= \left(0 + 0 + 2 + 6x + 12x^2 + 20x^3 + \dots\right)$$

$$= \boxed{2 + 6x + 12x^2 + 20x^3 + \dots}$$

Q. 40) Assume $\sqrt{1+x} \approx 1 + \frac{x}{2}$. Estimate the error if $|x| < 0.01$

* Use Maclorten series $\sqrt{1+x}$, $c = 0$

$f(x) = \sqrt{1+x}$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$

$f'''(x) = \frac{3}{8}(1+x)^{-2.5}$

$f(0) = 1$, $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$

$f'''(0) = \frac{3}{8}$

$$\sqrt{1+x} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{x^3}{16} - \cdots$$

$$\text{Error} < \left| -\frac{1}{8}x^2 \right| = \frac{x^2}{8} < \frac{(0.01)^2}{8} = 1.25 \times 10^{-5}$$

$E < 1.25 \times 10^{-5}$ by (A.S.E.T)

* **Taylor Th:-**

f, f', f'', f''' cont. on $[a,b]$, Then, $\exists$ a number $c \in (a,b)$ s.t:-

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(b-a)^n + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}}_{\text{الباقي}}$$

* **Note:-** MVT:- f cont on $[a,b]$, f dif on (a,b) $\Rightarrow \exists\, c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

MVT is special case from Taylor Th.

$\to f(b) = f(a) + f'(c)(b-a)$

$f(b) - f(a) = f'(c)(b-a)$

$\hookrightarrow f'(c) = \dfrac{f(b) - f(a)}{b-a}$

* Replace b by $X \Rightarrow$

$$f(x) = \underbrace{\left[f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''}{3!}(x-a)^3 + \dots \frac{f^{(n)}(a)}{n!}(x-a)^n \right]}_{P_n(x)\text{ "Poly. of degree } n\text{"}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}}_{R_n(x)\text{ "Remanider"}} \to c \in (a, x)$$

center

$f(x) = P_n(x) + R_n(x) \Rightarrow$ Taylor formula (T.F)

$P_n(x) \approx f(x)$ with error $= |R_n|$

* **Remark:-** (Convergence of Taylor series)

If $\lim_{n\to\infty} R_n(x) = 0 \quad \forall\, x \in I$, then

→ Taylor series generated by f at $x = a$

Converges to f on I

That is:- $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

* **Exp:-** Show that Taylor series generated by $f(x) = e^x$ at $x=0$ Converges to $f(x)$ $\forall x$

Taylor series of $f(x) = e^x$ at $x=0$ is Maclurine series;

$$f(x) = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}}_{P_n(x)} + \underbrace{R_n(x)}_{R_n(x)}$$

* T.F $\Rightarrow P_n(x) + R_n(x)$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$$

$f^{(n+1)}(c) \to e^c$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c\, x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} R_n(X) = \lim_{n\to\infty} \frac{e^c X^{n+1}}{(n+1)!}$$

$$= e^c \lim_{n\to\infty} \frac{X^{n+1}}{(n+1)!} \to \text{Zero}$$

$$= \text{Zero}$$

Hence, Maclurine series converges to e^X

$$e^X = 1 + X + \frac{X^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

$$e^1 = 1 + 1 + \frac{1^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{1^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$e \approx 2.718 \ldots$

* Th) (The Remainder Estimation Th)

(T.F) $f(x) = P_n(x) + R_n(x)$

$$\hookrightarrow \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Assume $\left| f^{(n+1)}(c) \right| \leq M$ for all $t \in (a, x)$

رقم موجب

Upper Bound for $f^{(n+1)}$

Then, Remainder $= \left| R_n(x) \right| \leq \underbrace{\frac{M}{(n+1)!} \left| x - a \right|^{(n+1)}}_{*}$

If * holds for all n, then the Taylor series generated by $f(x)$ convergece to $f(x)$.

*Exp:- Show that the Maclurine Series for $\sin x$ converges to $\sin x$ $\forall x$

by (T.F) $= P_{2n+1}(x) + R_{2n+1}(x)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \frac{f^{2n+2}(c)}{(2n+2)!} x^{2n+2}$$

$f = \sin x$

$f' = \cos$

$f'' = -\sin x$

$f''' = -\cos x$

$\vdots$

$$|R_{2n+1}| = \left| \frac{f^{2n+2}(c)}{(2n+2)!} x^{2n+2} \right|$$

$$|R_{2n+1}| \leq \frac{1}{(2n+2)!} |x^{2n+2}|$$

$|f^{2n+2}| \leq 1$ ← M upper bound

$$0 \leq |R_{2n+1}(x)| \leq \frac{|x^{2n+2}|}{(2n+2)!}$$

$\lim_{n\to\infty}$ لجميع الأطراف

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \left| \frac{x^{(2n+2)}}{(2n+2)!} \right| = 0$

by S.T $\lim_{n\to\infty} R_{2n+1} = 0$ ∴ converges $0 \leq \lim R_{2n+1}(x) \leq 0$